Combinatorial aspects in recurrent seqences over finite alphabets

Mihai Prunescu

Bucharest, September 8, 2013

MOTTO

.... wir haben die Kunst, damit wir nicht an der Wahrheit zugrunde gehen .....

Friedrich Nietzsche

Definition

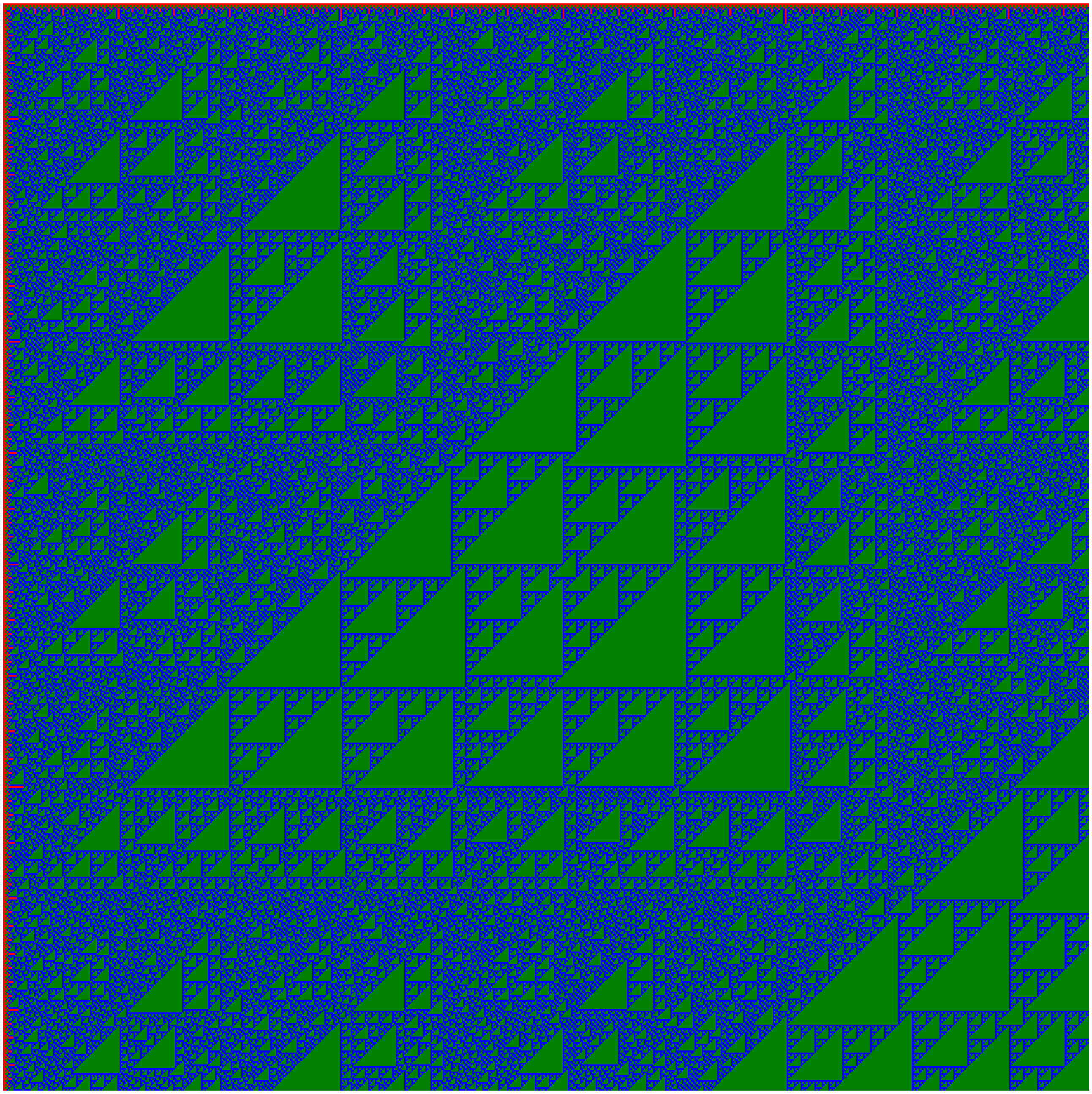
$(A, f, 1)$: A finite, $f : A^3 \to A$, $1 \in A$

Recurrent double sequence $(a(i,j))$:

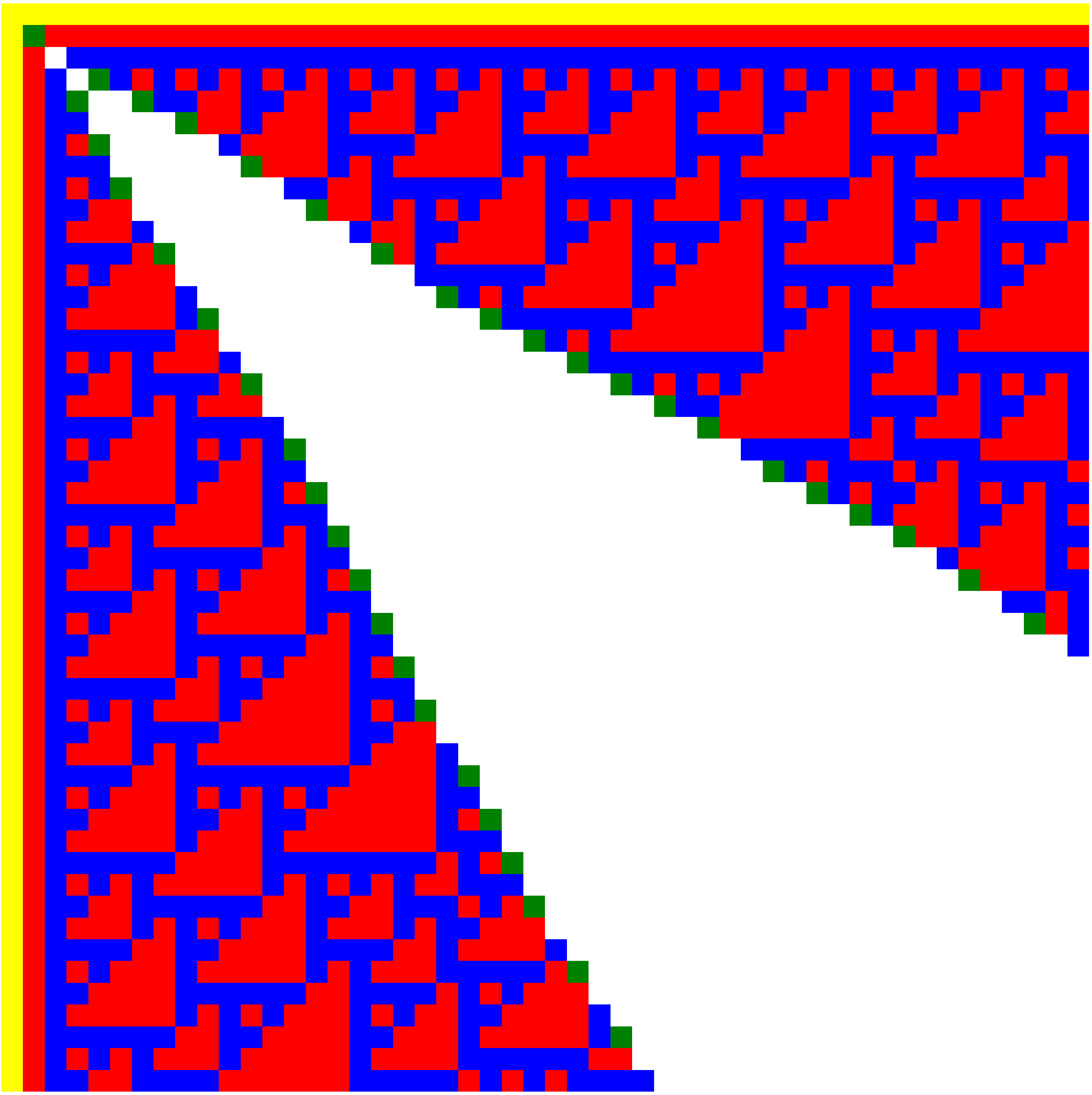
- $\forall i \;\; \forall j \;\; a(i,0) = a(0,j) = 1$

- $i > 0 \;\wedge\; j > 0$:

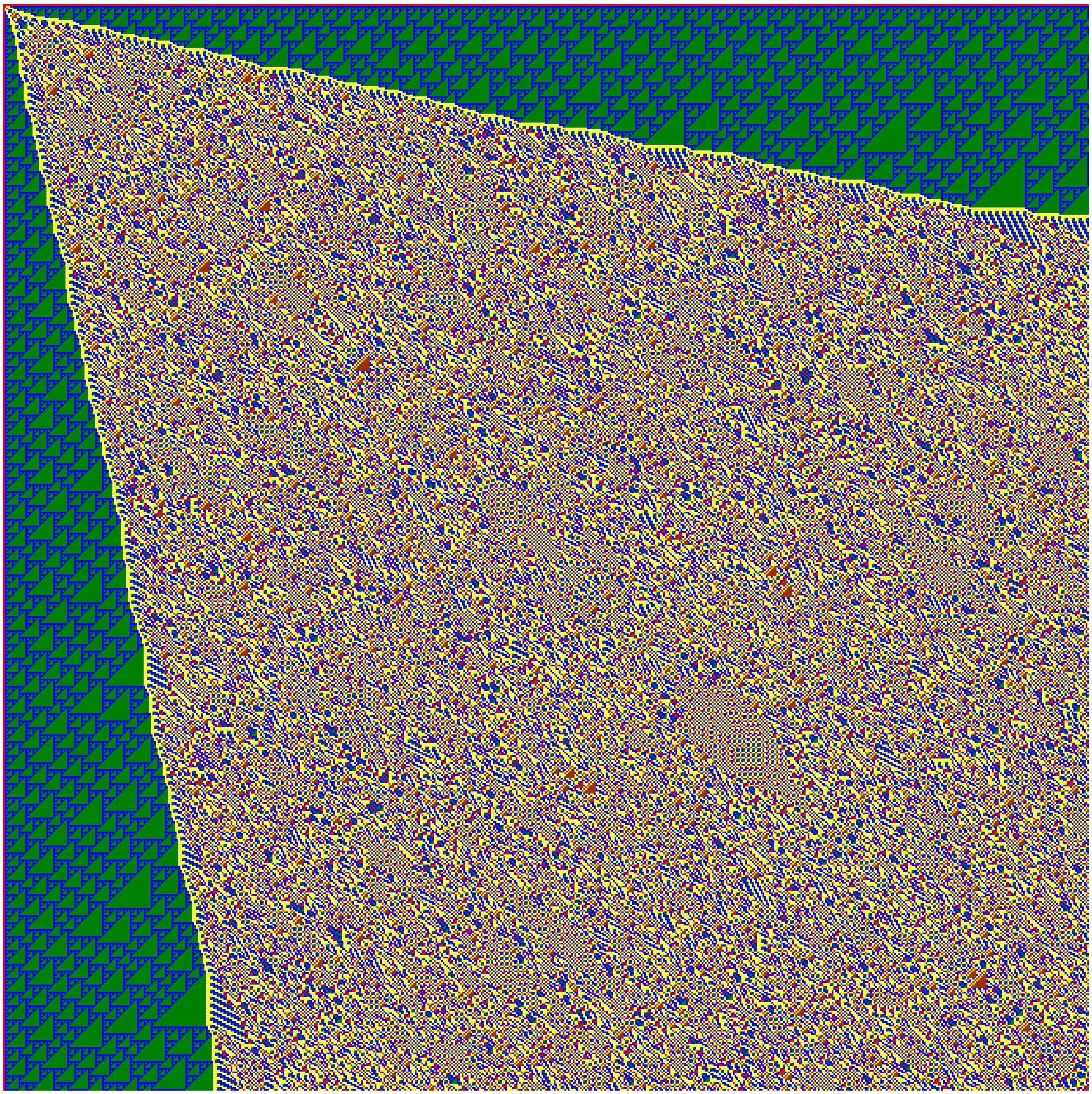
$$a(i,j) = f(a(i-1,j), a(i-1,j-1), a(i,j-1))$$



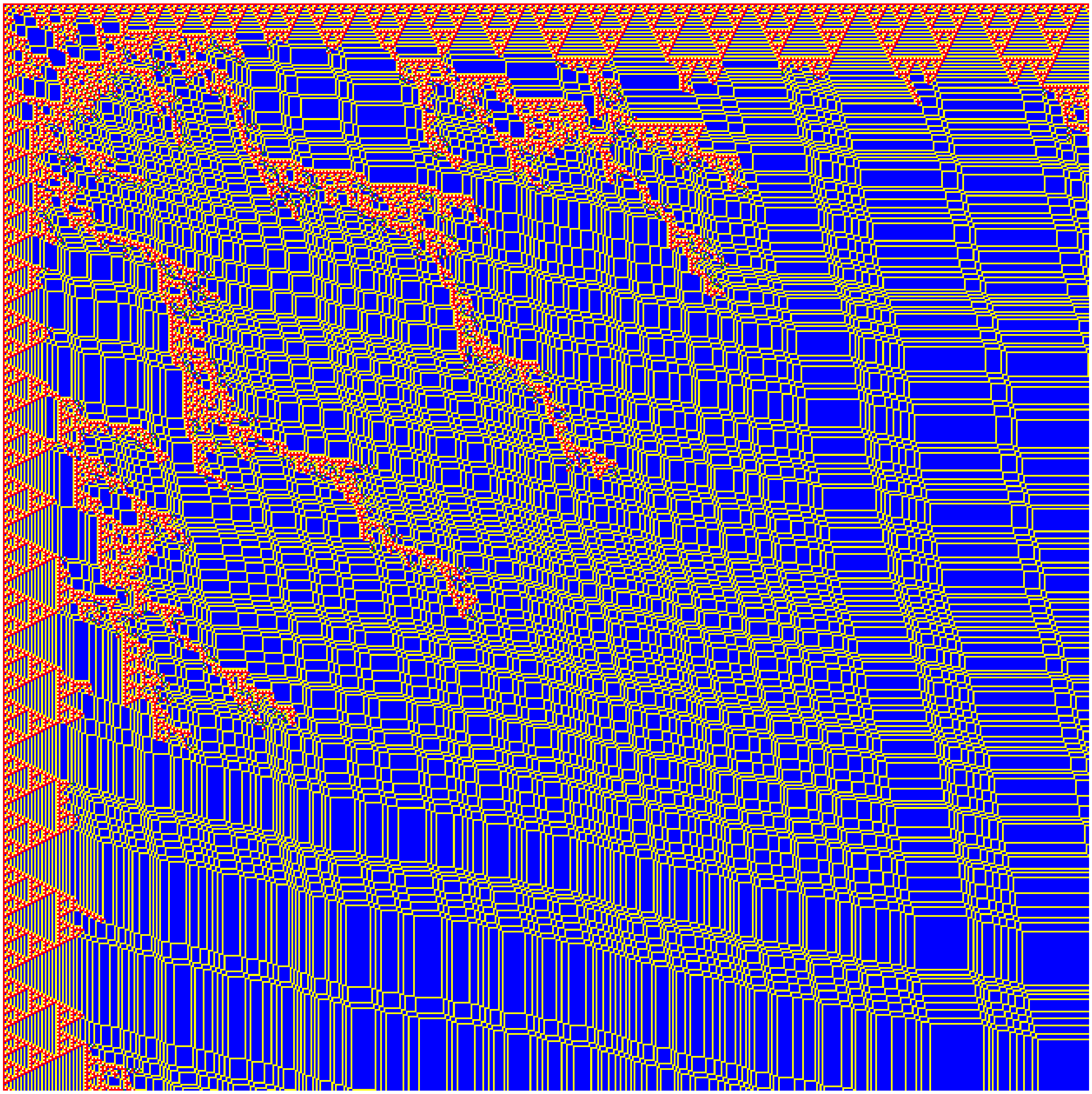
$(\mathbb{F}_5, 4x^2y^4z^2 + 4x^4y^3 + 4y^3z^4 + 2xy^2z + 3, 1)$



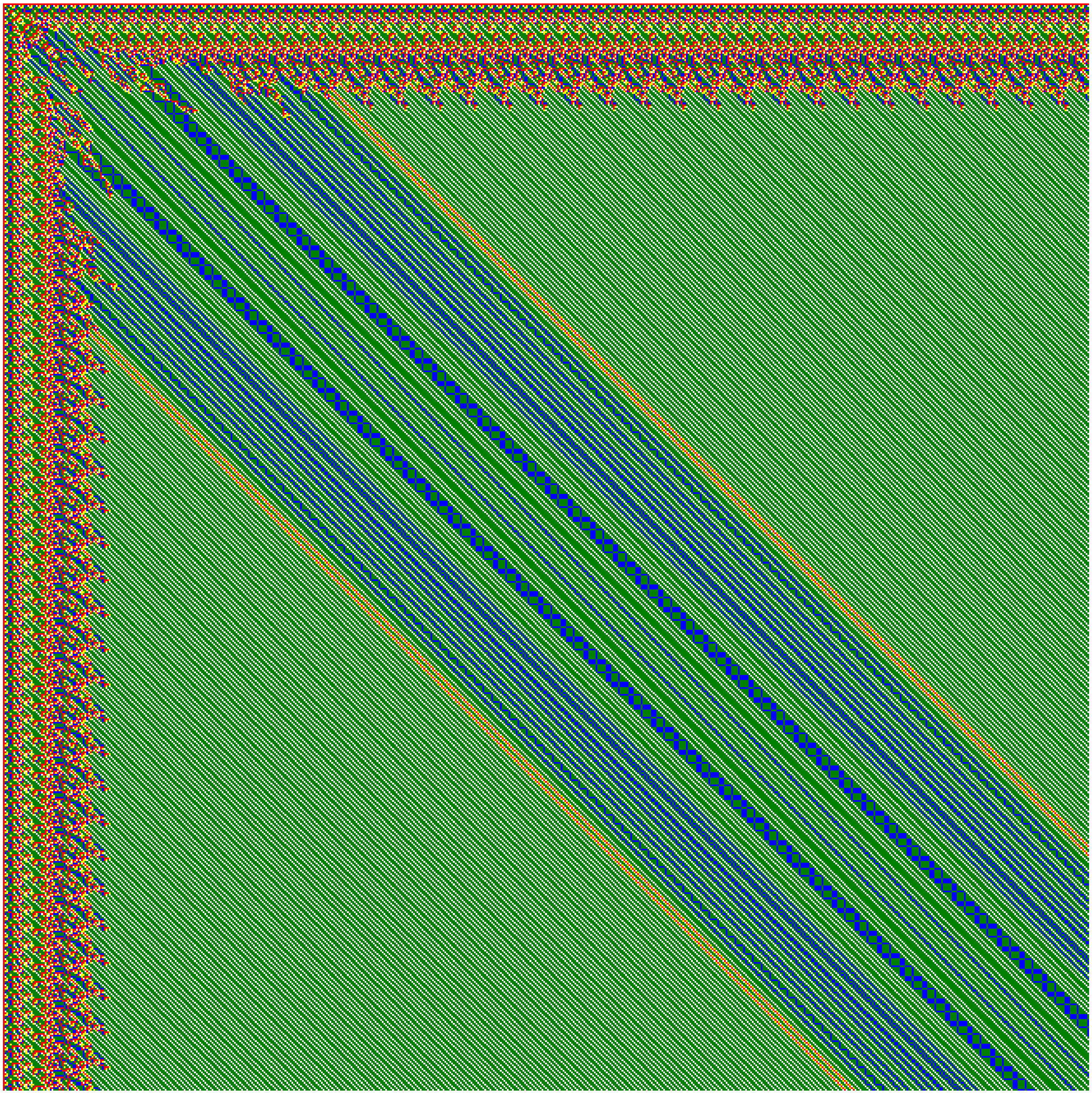
$(\mathbb{F}_5, 4x^4z^4 + 4x^2y^2 + 4y^2z^2 + 4y^2, 2)$



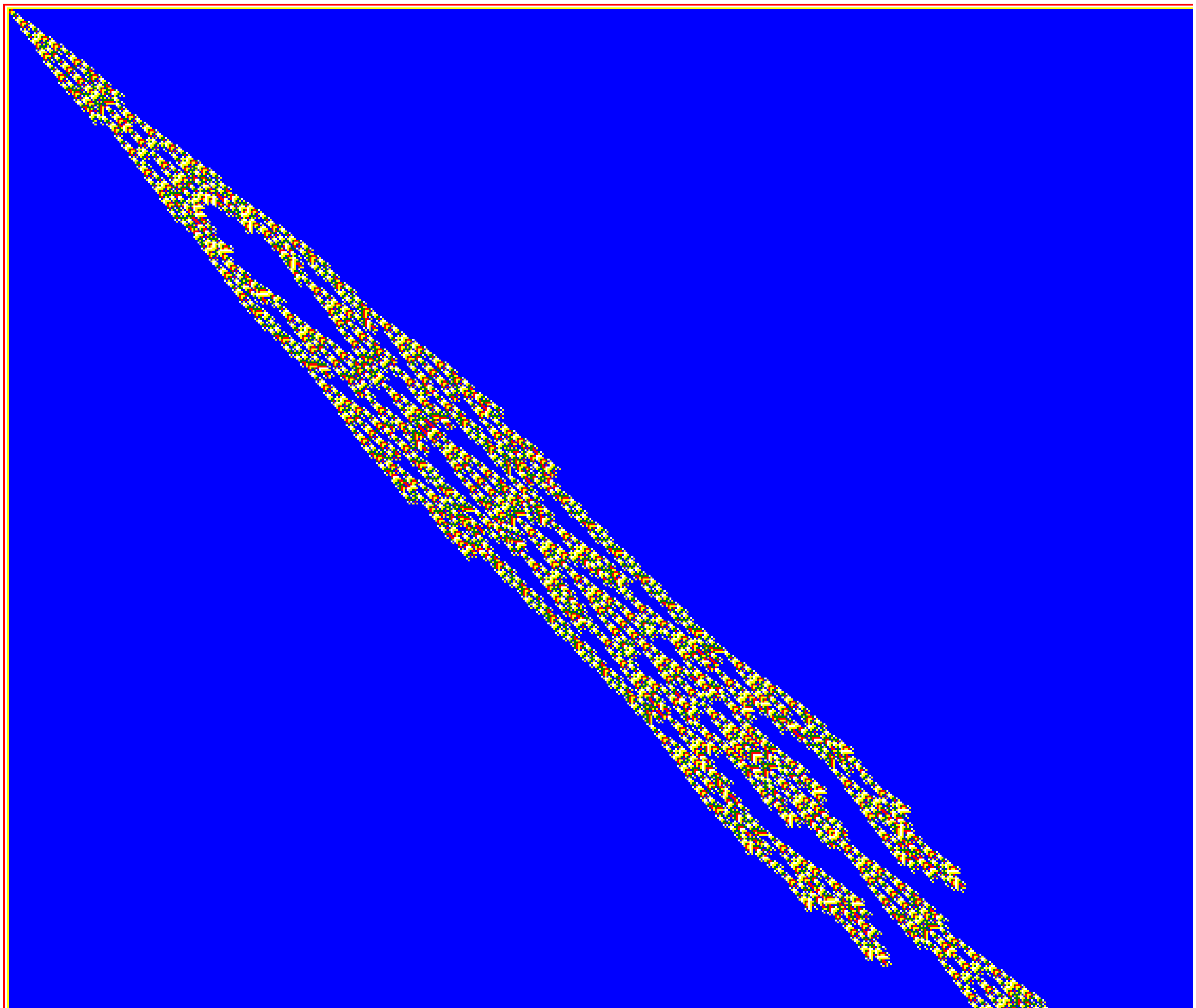
$(\mathbb{F}_5, 3x^4z^4 + 3x^2y^2 + 3y^2z^2 + 2x^3yz^3 + 1, 1)$



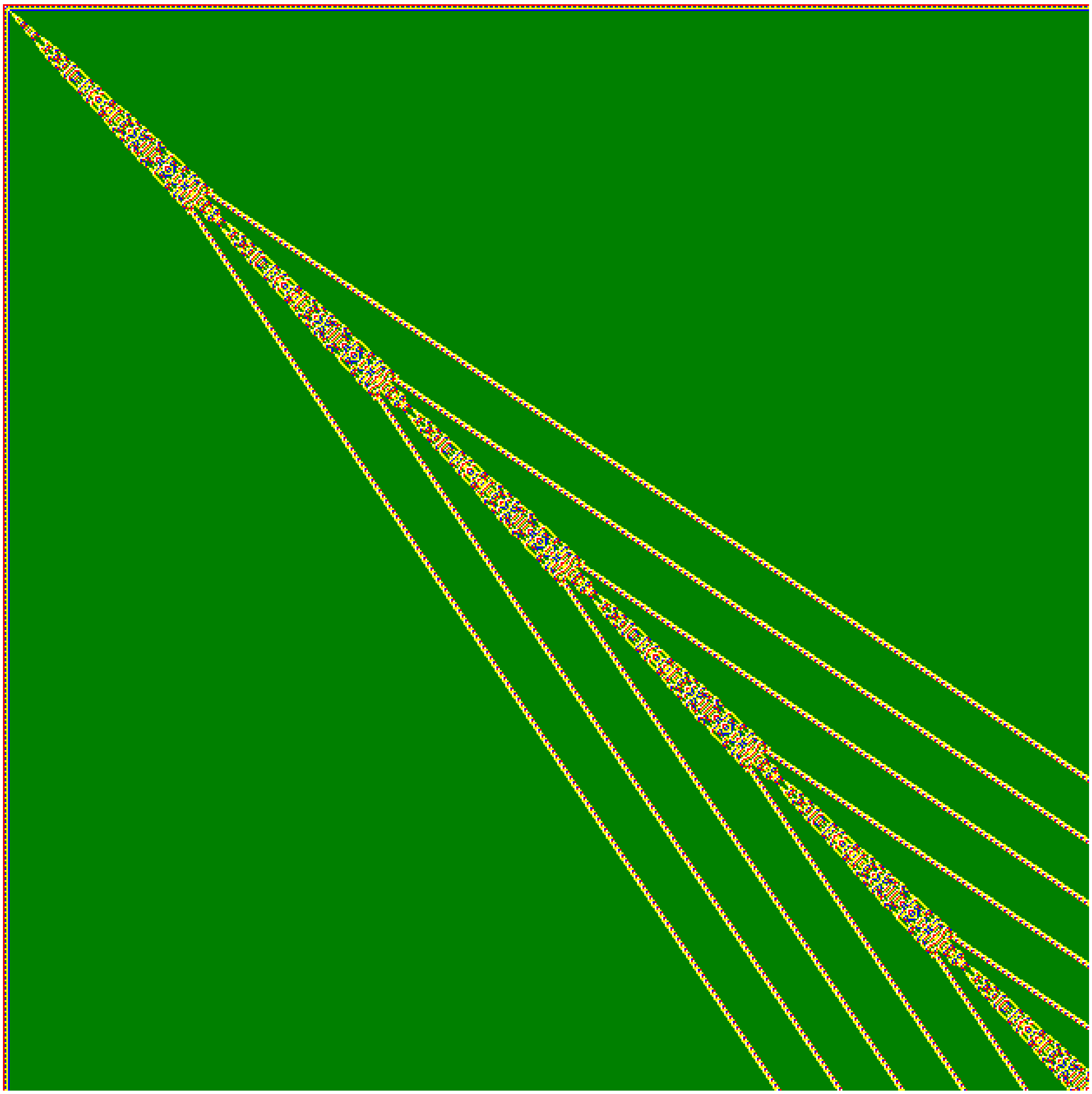
$(\mathbb{F}_5, 4x^4z^4 + 4x^2y^2 + 4y^2z^2 + 4x^3y^2z^3 + 2, 1)$



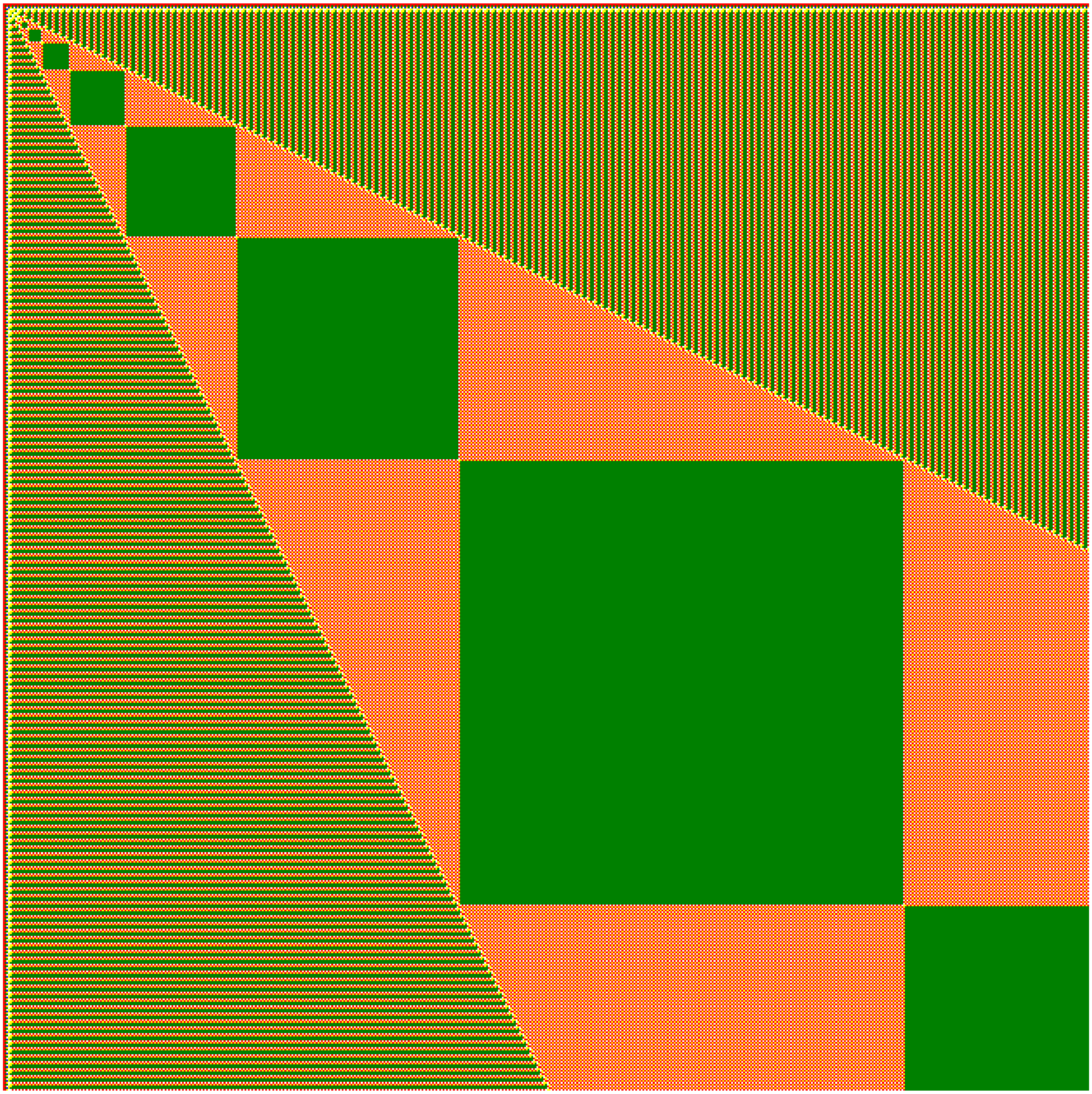
$(\mathbb{F}_5, 2x^3y^3z^3 + 2x^2 + 2z^2 + 4xy^3z + 4, 1)$



$(\mathbb{F}_5, x^2y^3z^2 + x^4y^2 + y^2z^4 + 3x^3y^3z^3 + 4, 1)$



$(\mathbb{F}_5, 2x^3y^2z^3 + 2x^3y^3 + 2y^3z^3 + 3x^4z^4 + 1, 1)$



$(\mathbb{F}_5, 3x^3y^2z^3+3x^3y^3+3y^3z^3+4x^2y^2z^2+4, 1)$

Turing Completeness

$(A, f : A^2 \to A, 0, 1)$

$a(i,j) = f(a(i,j-1), a(i-1,j))$

Theorem 1 $\forall$ (M, w) *Turing Machine with input* $\exists$ $\mathfrak{A} = (A, f, 0, 1)$ *finite, commutative, so that:*

$(a(i,j))$ *ultimately zero*

$$\Longleftrightarrow$$

M *stops with empty band, without having ever been left from the start cell.*

M. P: *Undecidable properties of the recurrent double sequences*. Notre Dame Journal of Formal Logic, 49, 2, 143 - 151, 2008.

a, b, c, d letters

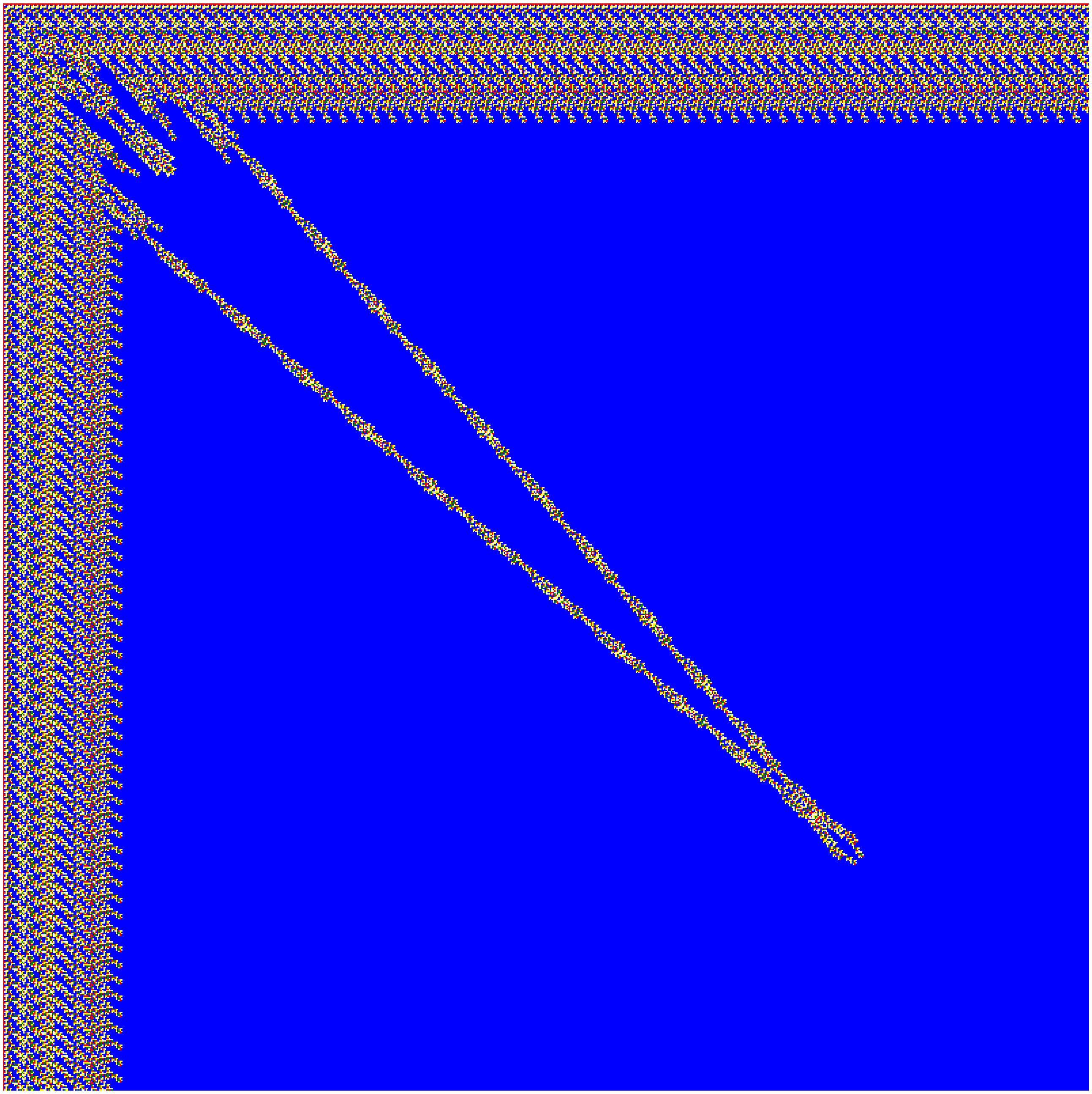
z state

$\delta = (c, z)$ new letter

$$
\begin{array}{ccc}
 & & b \\
 & \delta & (\delta, b) \\
a & (a, \delta) & d
\end{array}
$$

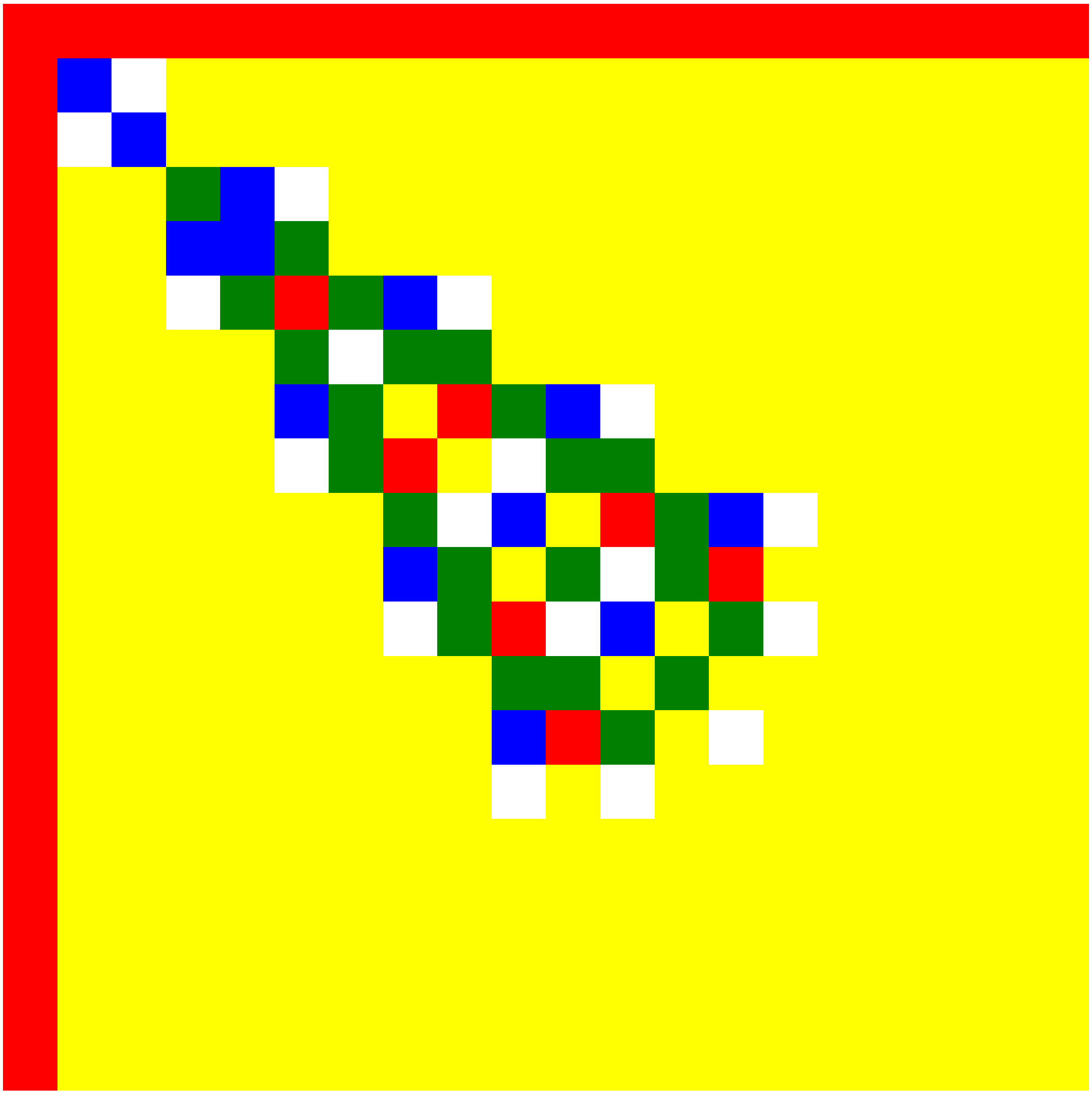
To construct a commutative structure, one needs 8 diagonals in stead of only 2.

"Stopping Computation 1", 625×625



$(\mathbb{F}_5, 4x^4z^4 + 4xy^3 + 4y^3z + 4xy^3z + 4, 1)$

"Stopping Computation 2", 20×20



$(\mathbb{F}_5, x^4z^4 + x^2y^4 + y^4z^2 + 2xyz + 3, 1)$

Selfsimilar double sequences

$(\mathbb{F}_q, f(x,y,z) = x + my + z, 1)$

$F = (a(i,j) \mid 0 \leq i,j < p),\ q = p^s$

$\varphi(x) = x^p$ Frobenius' Automorphism

$G_d = (a(i,j) \mid 0 \leq i,j < p^d)$

Theorem 2

$$G_d = \varphi^{d-1}(F) \otimes \varphi^{d-2}(F) \otimes \cdots \otimes \varphi(F) \otimes F$$

If $\mathbb{F}_q = \mathbb{F}_p$, then $G_d = F^{\otimes d}$. Substitution: start with 1 and apply rules of type:

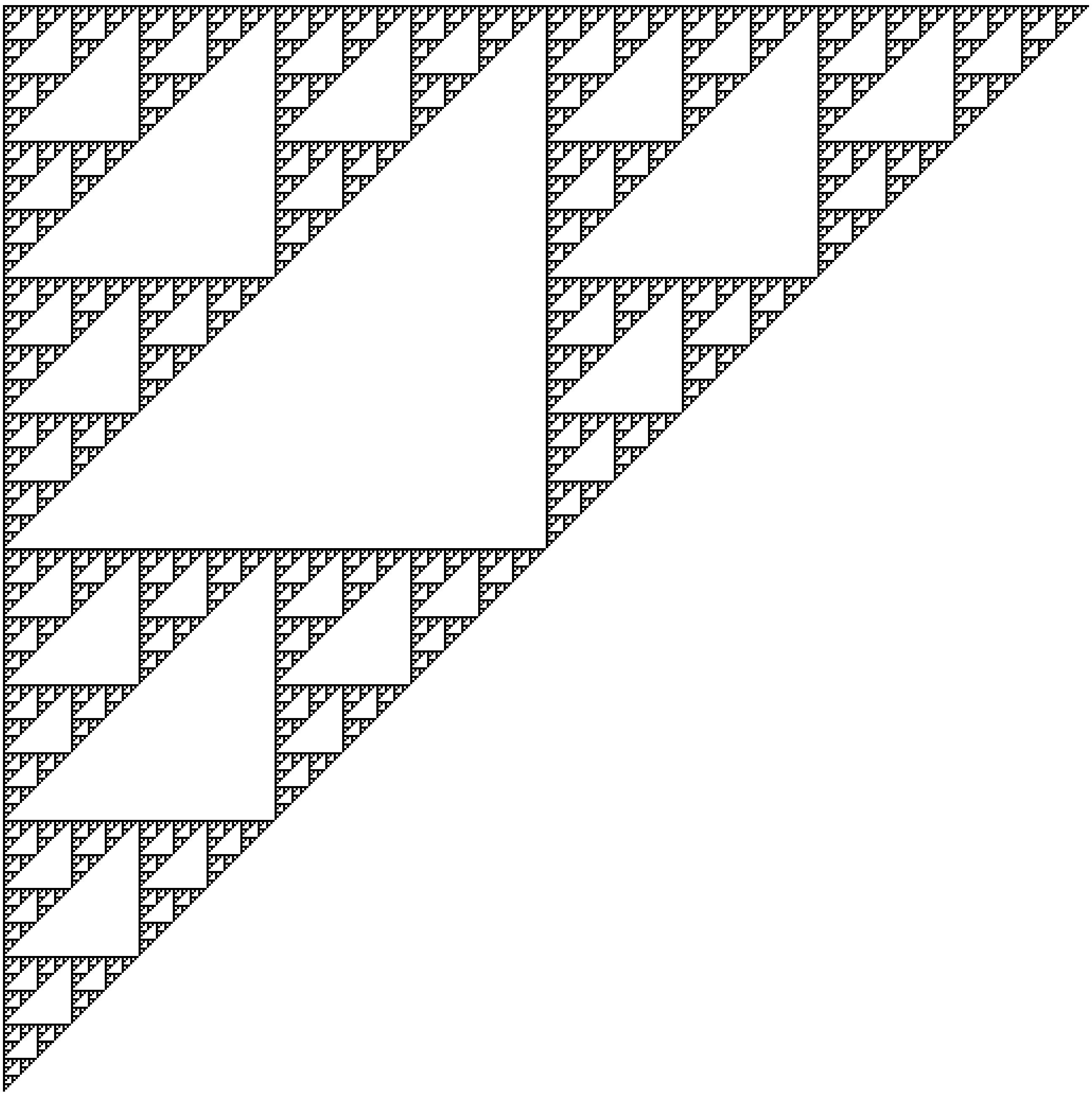
$$element \rightarrow matrix$$

$$a \rightarrow aF$$

The sequence f matrices (G_d) converges to a self-similar *fractal*.

M. P: *Self-similar carpets over finite fields*. European Journal of Combinatorics, 30, 4, 866 - 878, 2009.

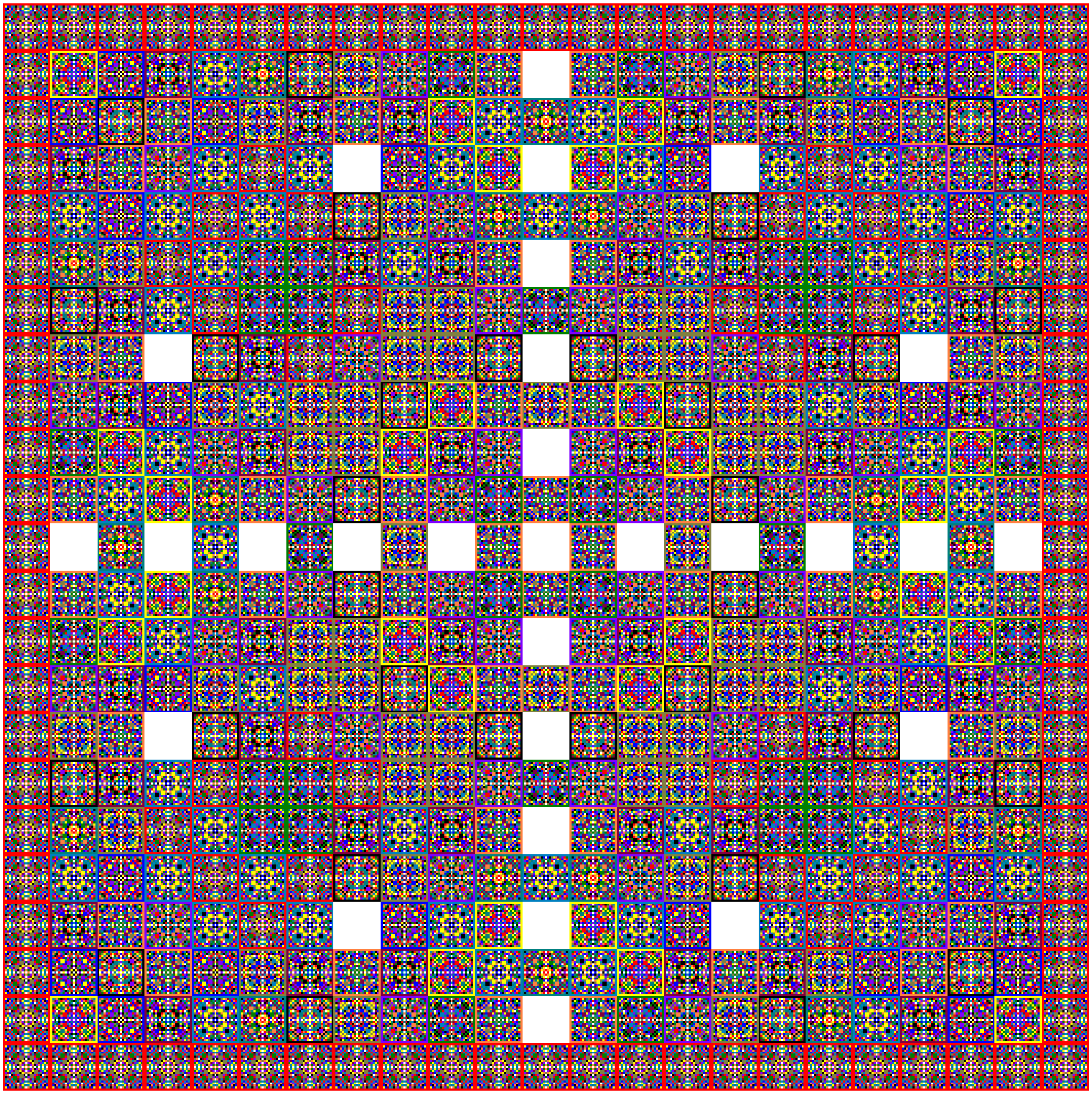
Pascal's Triangle mod 2



$(\mathbb{F}_2, x+z, 1)$, $d = 9$

$$1 \to \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad 0 \to \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Lakhtakia - Passoja Carpet mod 23



$(\mathbb{F}_{23}, x + y + z, 1)$, $d = 2$

G_2: $\forall\, k \in \mathbb{F}_{23}$ Color$(k) =$ Color$(23 - k)$

Substitution

[*element* $\to$ *matrix*] $\leadsto$ [*matrix* $\to$ *matrix*]

$x \geq 1$ basic granulation

$s \geq 2$ scaling, $y = xs$

$\mathcal{X} \subset A^{x \times x}$ finite

$\mathcal{Y} \subset A^{y \times y}$ finite

$\forall\ Y \in \mathcal{Y} \quad Y = (X(i,j) \in \mathcal{X} \mid 0 \leq i,j < s)$

$\Sigma : \mathcal{X} \to \mathcal{Y}$ rule of substitution

$X_1 \in \mathcal{X}$ start symbol

$(\mathcal{X}, \mathcal{Y}, \Sigma, X_1)$ system of substitutions

$S(1) = X_1$, $S(n) = \Sigma^{n-1}(X_1)$

Expansive systems of substitutions

$(\mathcal{X}, \mathcal{Y}, \Sigma, X_1)$ expansive, if

$\Sigma(X_1) = (X(i,j) \in \mathcal{X}) \models X(0,0) = X_1$

Lemma 3 $(\mathcal{X}, \mathcal{Y}, \Sigma, X_1)$ *expansive. Then for all* $n > 0$ *is the matrix* $S(n)$ *the* $xs^{n-1} \times xs^{n-1}$ *left upper corner of the matrix* $S(n+1)$*.*

$$S(n+1) = \begin{pmatrix} S(n) & U \\ V & W \end{pmatrix}$$

Let $T \in A^{wx \times zx}$ be a matrix.

Definition:

$\mathcal{N}_x = \{K \in A^{2x \times 2x} \mid K$ occurs in T and starts in some $(kx, lx)\}$

Theorem 4 $(A, f,$ *Margins* $) \rightsquigarrow R$

$(A, \mathcal{X}, \mathcal{Y}, \Sigma, X_1)$, $x \to sx$, $\rightsquigarrow S$

$R(n) := (a(i,j) \,|\, 0 \le i, j < xs^{n-1})$

If *there is some* $m > 1$*, so that :*

- $R(m) = S(m)$

- $\mathcal{N}_x(S(m-1)) = \mathcal{N}_x(S(m))$

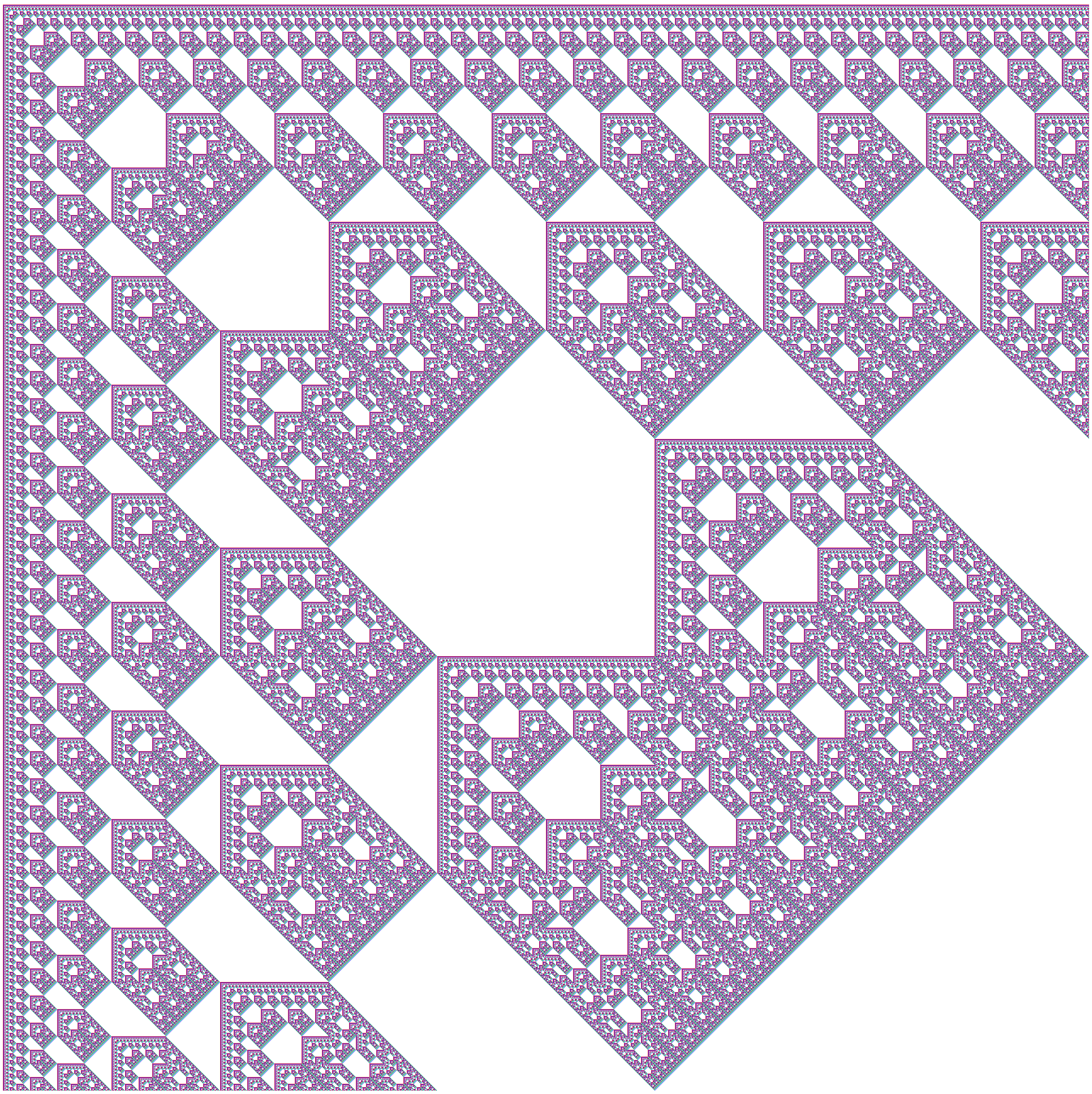
- $S \,|\, (i = 0) = R \,|\, (i = 0)$

- $S \,|\, (j = 0) = R \,|\, (j = 0)$

Then $R = S$.

M. P: *Recurrent double sequences that can be produced by context-free substitutions.* Fractals, Vol 18, Nr. 1, 1 - 9, 2010.

Twin Peaks, 2560×2560.



$$\mathbb{F}_4 = \{0, 1, \epsilon, \epsilon^2 = \epsilon + 1\} = \{0, 1, 2, 3\}$$

$$(\mathbb{F}_4, y + \epsilon(x + z) + \epsilon^2(x^2 + y^2 + z^2), 1)$$

$$X_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

$$X_2 = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} X_2 & X_2 \\ X_5 & X_5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 2 & 0 \end{pmatrix}$$

$$X_3 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} X_3 & X_6 \\ X_3 & X_6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 1 & 2 & 1 & 0 \\ 1 & 2 & 0 & 2 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

$$X_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} X_7 & X_1 \\ X_1 & X_9 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 3 \\ 1 & 2 & 3 & 0 \end{pmatrix}$$

$$X_5 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} X_8 & X_1 \\ X_{10} & X_{11} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 3 & 0 & 1 & 2 \\ 0 & 2 & 0 & 3 \\ 2 & 0 & 0 & 0 \end{pmatrix}$$

$$X_6 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} X_{11} & X_{10} \\ X_1 & X_8 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 \end{pmatrix}$$

$$X_7 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} X_7 & X_7 \\ X_7 & X_7 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_8 = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} X_8 & X_7 \\ X_{12} & X_8 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 3 & 2 & 3 & 0 \end{pmatrix}$$

$$X_9 = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} X_9 & X_{12} \\ X_{12} & X_9 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 1 & 3 \\ 3 & 0 & 3 & 2 \\ 1 & 3 & 0 & 3 \\ 3 & 2 & 3 & 0 \end{pmatrix}$$

$$X_{10} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} X_9 & X_{10} \\ X_{10} & X_7 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}$$

$$X_{11} = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} X_{11} & X_{12} \\ X_7 & X_{11} \end{pmatrix} = \begin{pmatrix} 0 & 3 & 1 & 3 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

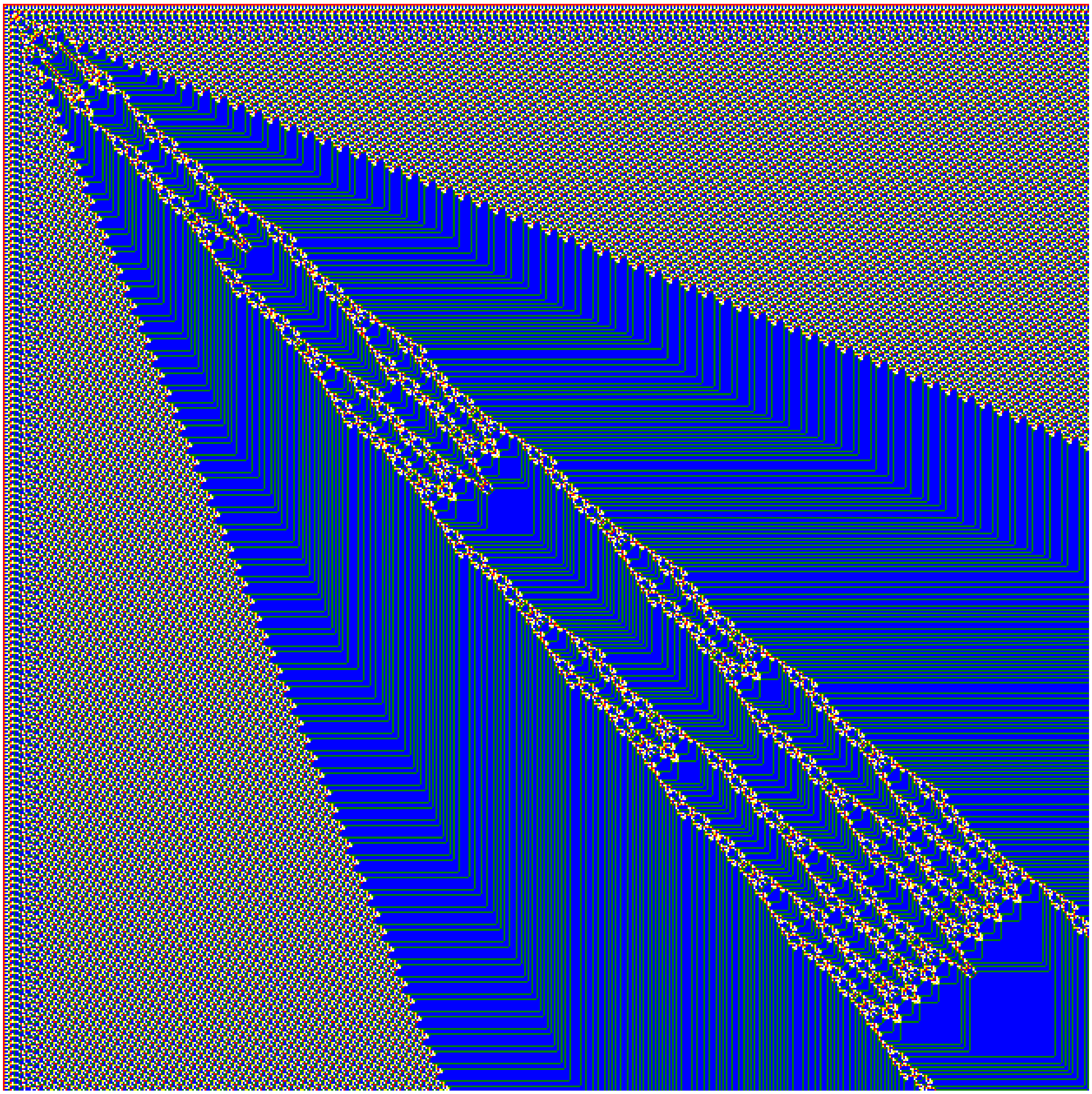
$$X_{12} = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} X_{13} & X_{14} \\ X_{15} & X_4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 2 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

$$X_{13} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} X_{13} & X_6 \\ X_5 & X_{10} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 2 & 2 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 2 & 0 \end{pmatrix}$$

$$X_{14} = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} X_3 & X_{14} \\ X_{11} & X_5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 1 & 2 & 0 & 2 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

$$X_{15} = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} X_2 & X_8 \\ X_{15} & X_6 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 2 & 3 & 0 \\ 1 & 0 & 0 & 2 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

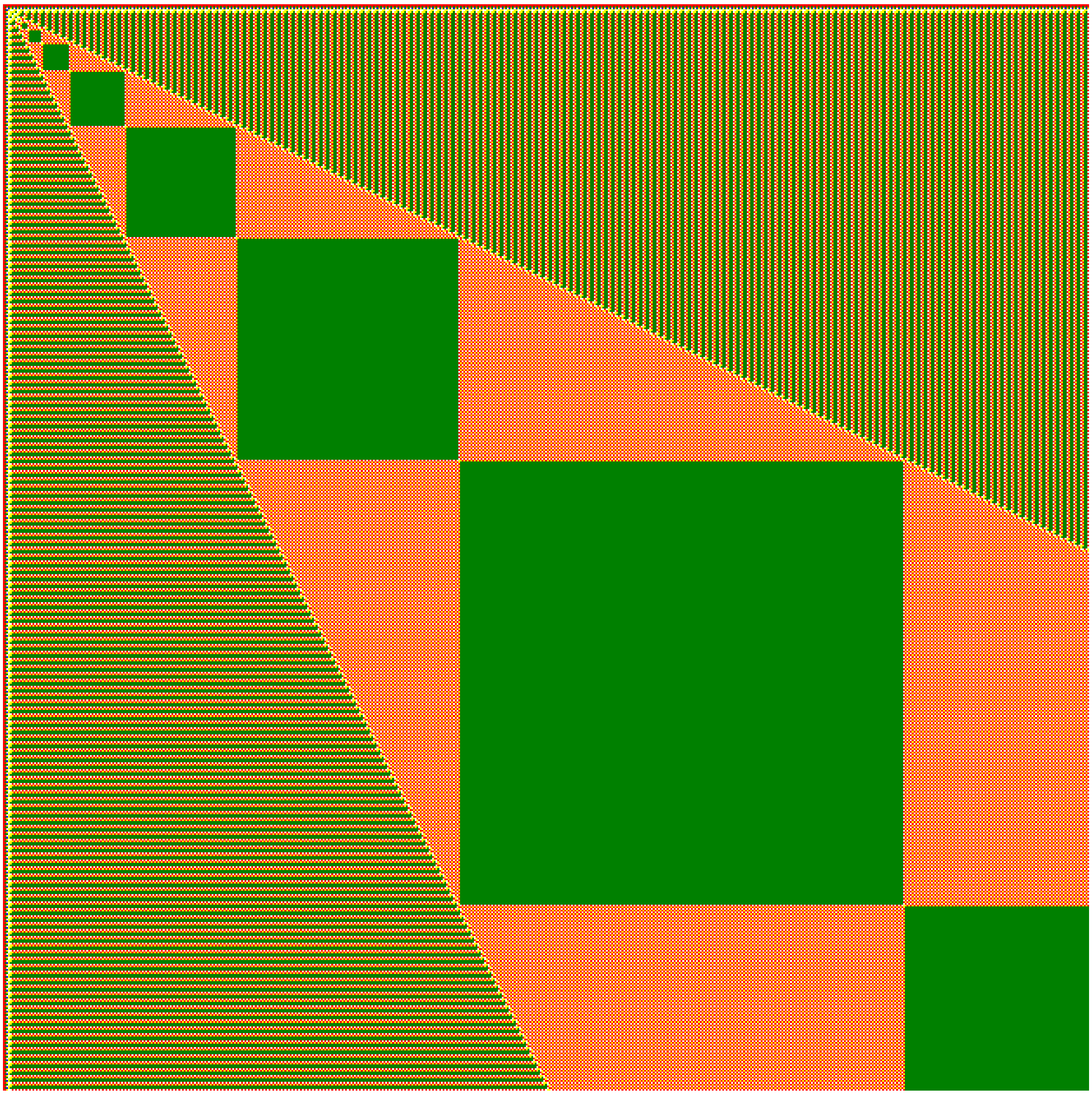
Ivy, 625×625



$(\mathbb{F}_5, x^3z^3 + x^4y + yz^4 + 2xyz + 4, 1)$

1802 rules $256 \to 512$

Square Root, 625×625



$(\mathbb{F}_5, 3x^3y^2z^3+3x^3y^3+3y^3z^3+4x^2y^2z^2+4, 1)$

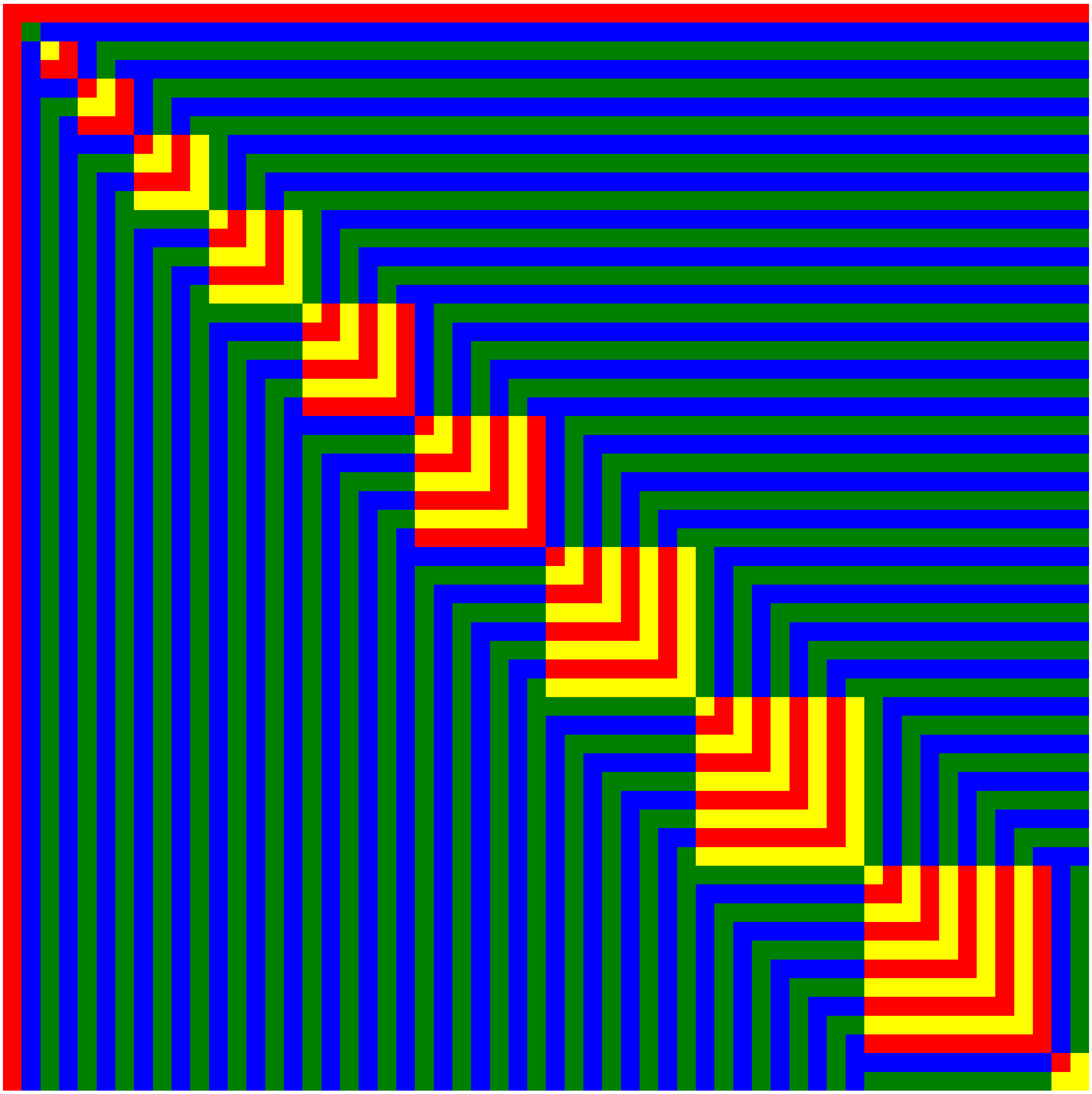
26 rules $8 \to 16$

Is every recurrent double sequence a substitution?

DEFINITELY NOT!

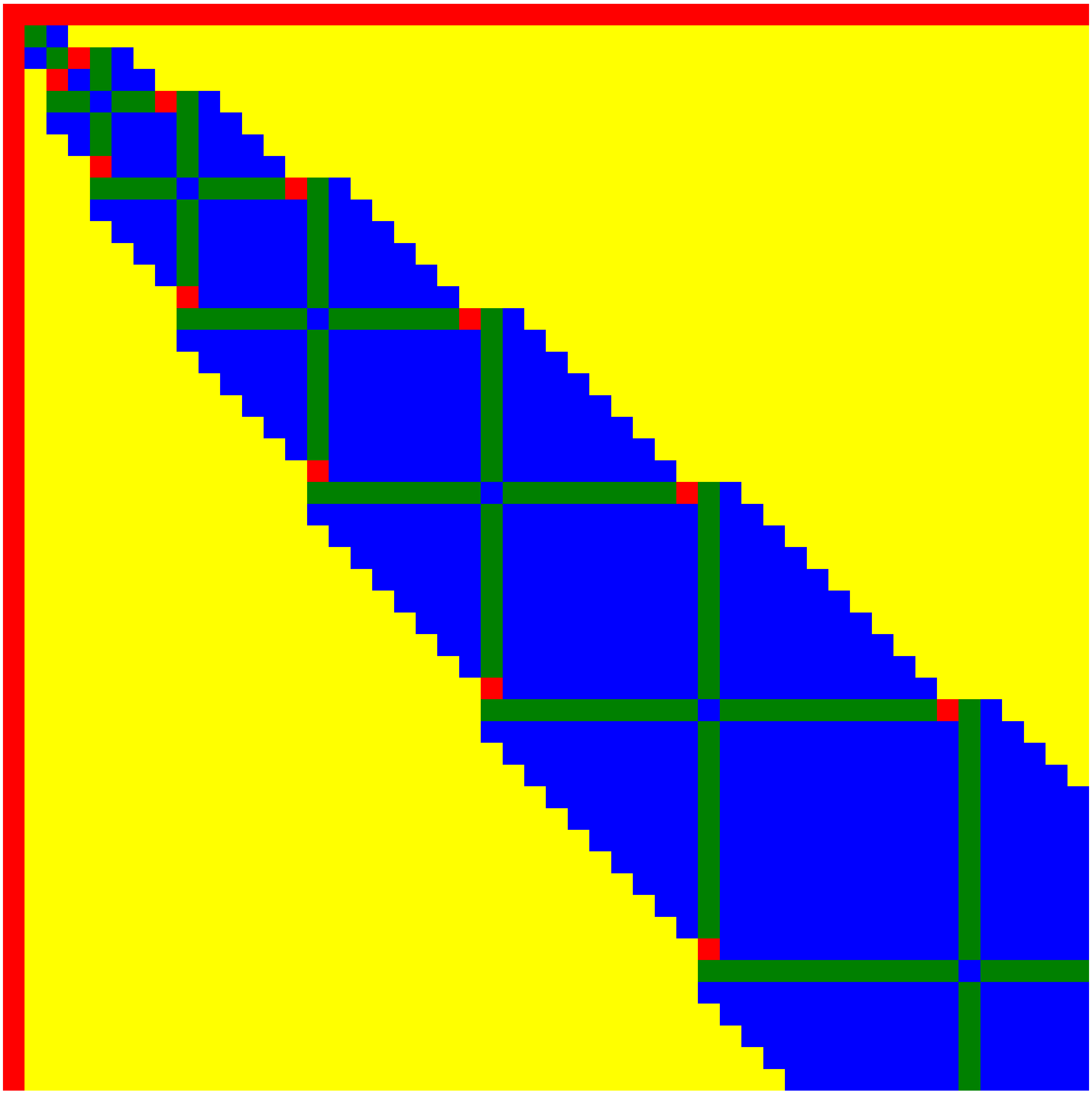
Some counterexamples interpret the set $\mathbb{N}$ of the natural numbers.

Stairway to Heaven, 58×58



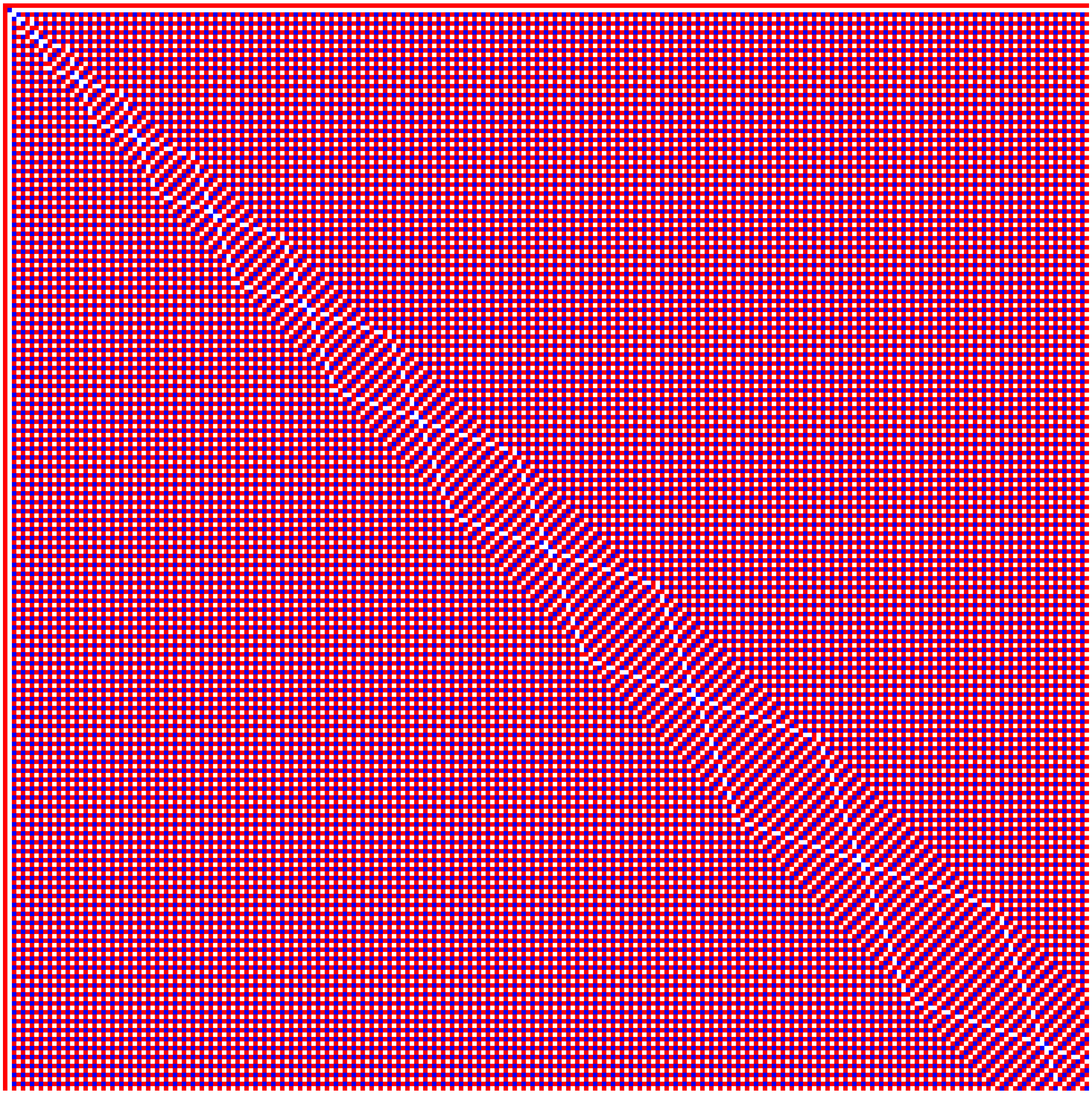
$(\mathbb{F}_5, 2x^3y^3z^3 + 2xy^2 + 2y^2z + y, 1)$

Second Stairway to Heaven, 50×50



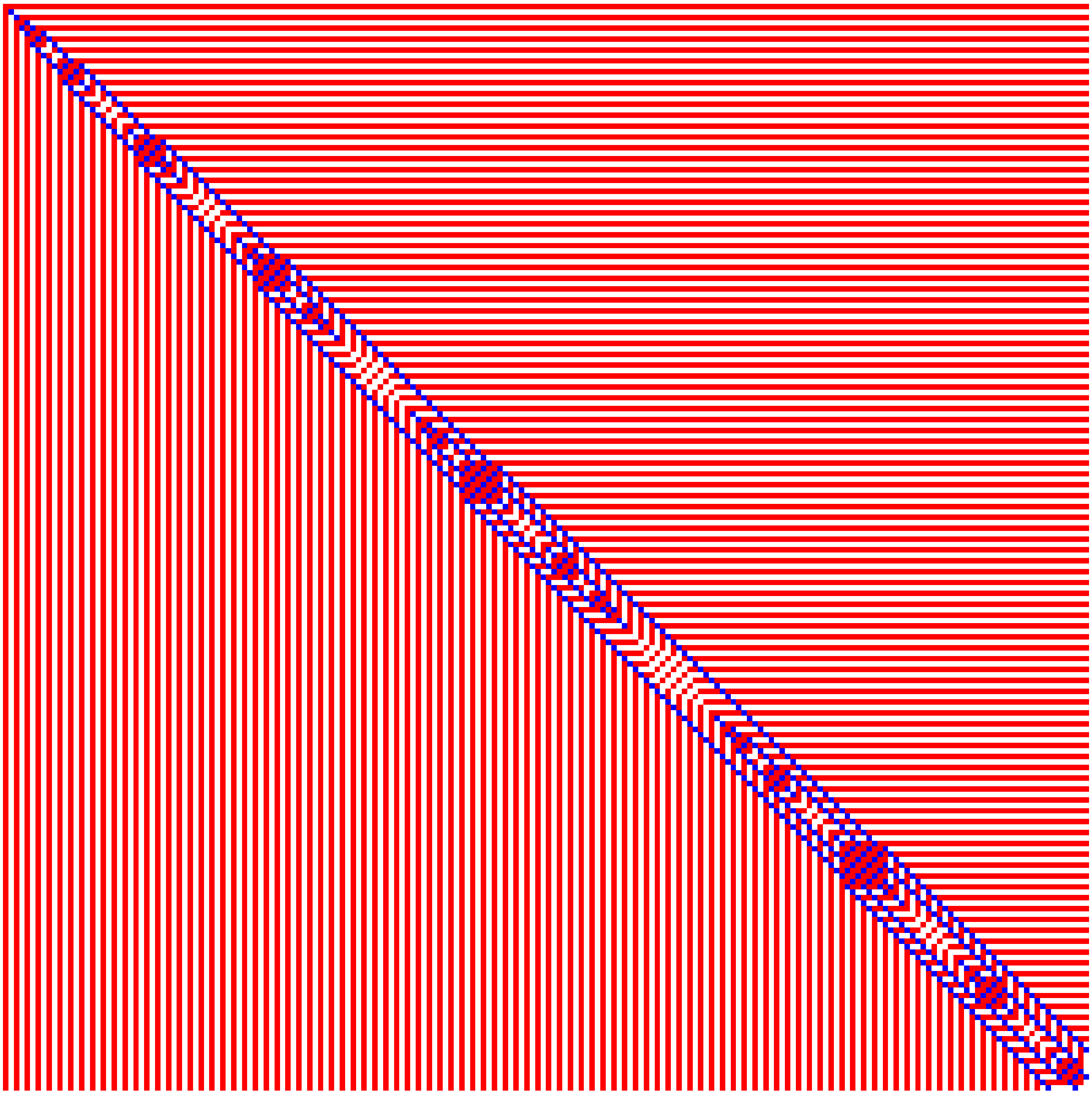
$(\mathbb{F}_5, 4x^3yz^3 + 4x^4y^2 + 4y^2z^4 + x^2y^2z^2 + 4, 1)$

Third Stairway to Heaven



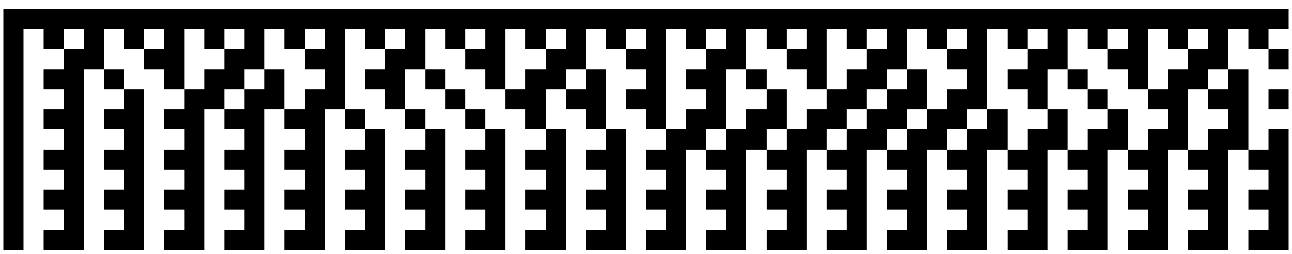
$$(\mathbb{F}_3, xy^2 + y^2z + xy + yz + x^2 + z^2 + 2x + 2z + 2, 1)$$

ORDINAL Stairway



$(\mathbb{F}_3, 2y^2 + x^2z + xz^2 + 1, 1)$

Minimal example of non-automatic recurrent double sequence



$(\mathbb{F}_2, 1 + x + z + yz, 1, 1)$, 64×12

The true reason of the minimal example

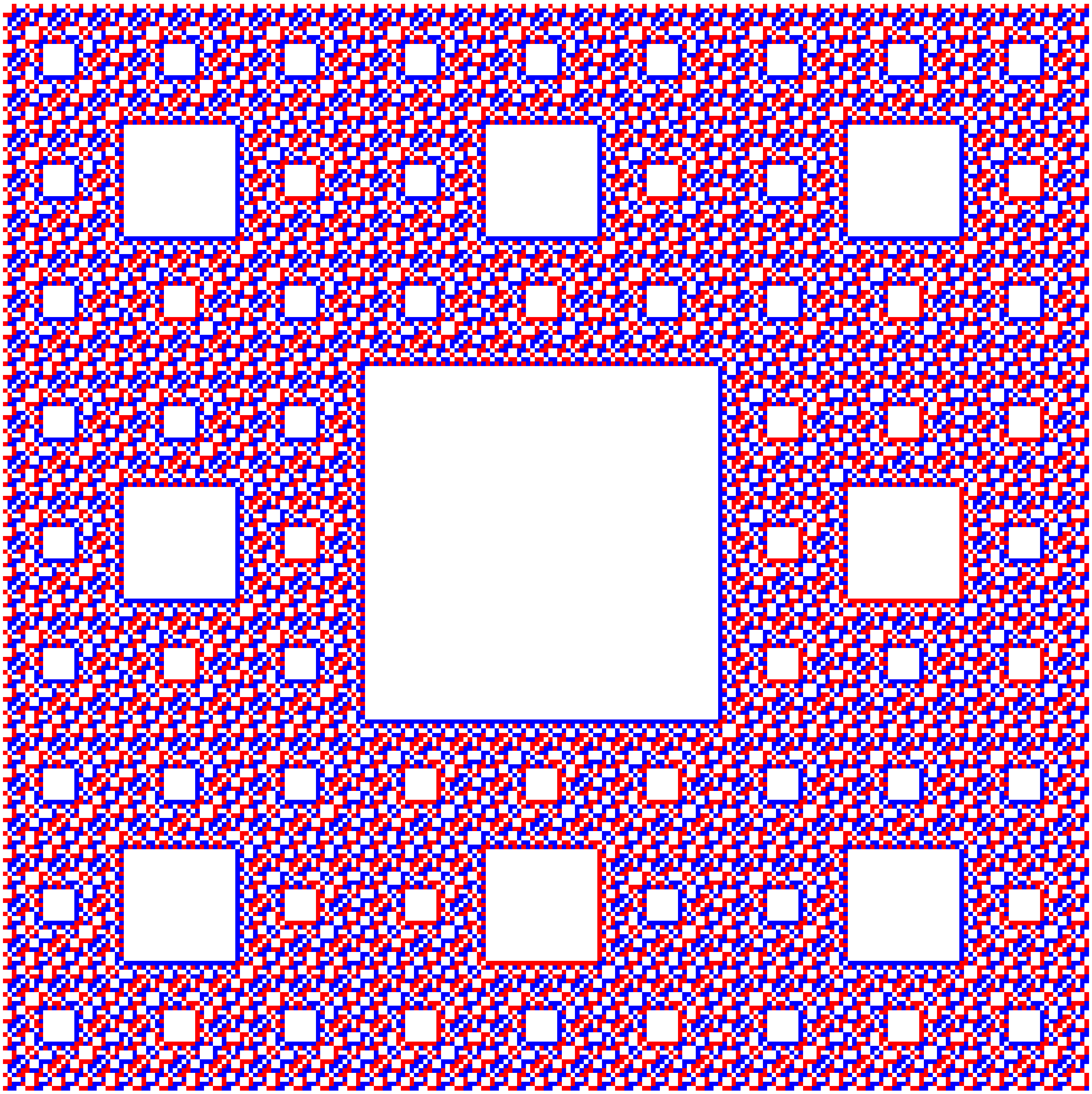
$(\mathbb{F}_2, x + y + yz, (01), 0)$, 128×10

Mihai Prunescu: A two-valued recurrent double sequence that is not automatic.

Margins as inputs

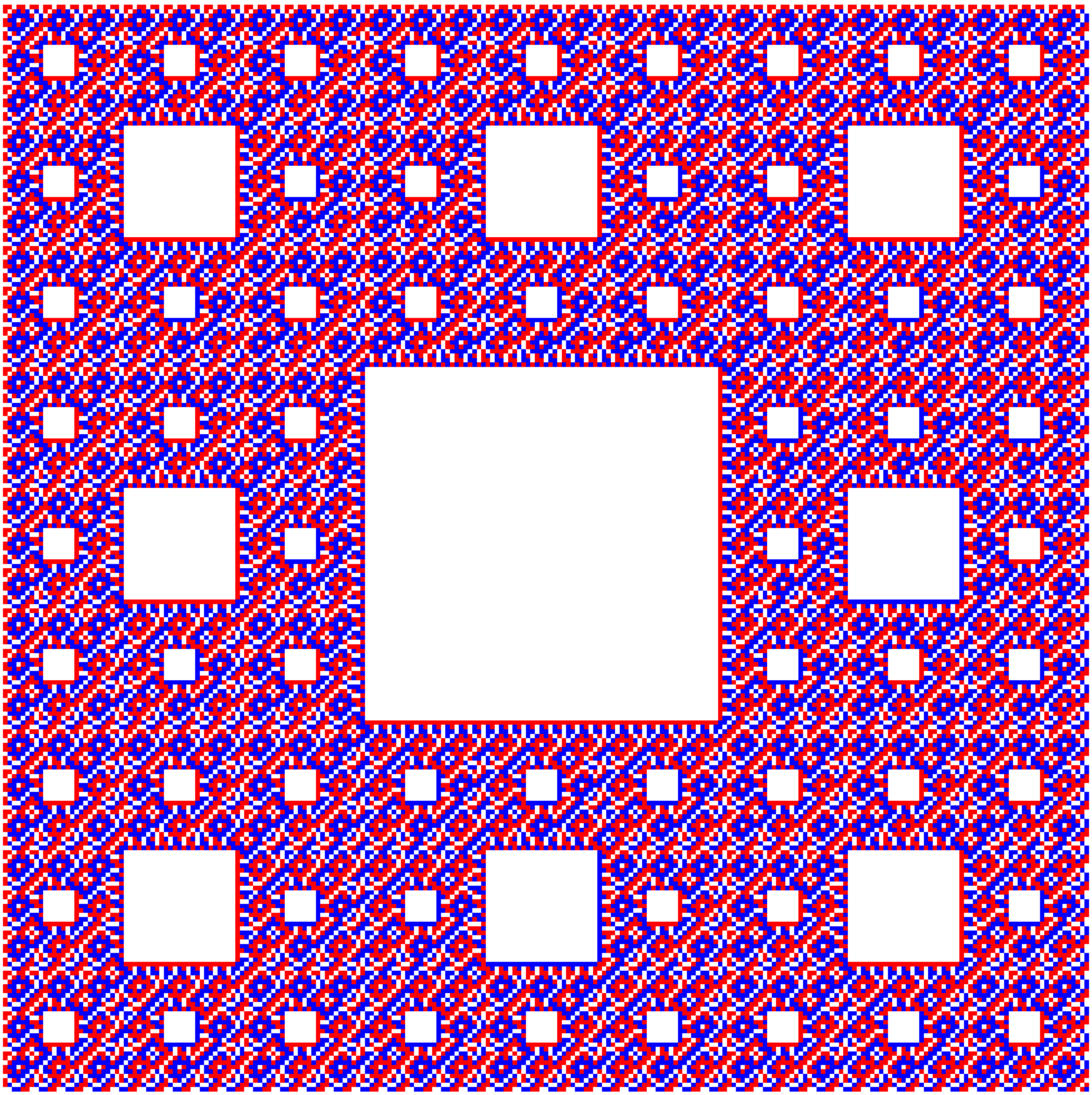
Periodic margins

$(\mathbb{Z}/3\mathbb{Z}, x + y + z, '001'), \; 243 \times 243$



23 rules $3 \to 9$

$(\mathbb{Z}/3\mathbb{Z}, x+y+z, '110'),\ 243 \times 243$



23 rules $3 \to 9$

Margins as input

Linear substitution

Thue - Morse Sequence

$(\{0,1\},\{0 \to 01, 1 \to 10\}, 0)$

$0110100110010110100101100110100 1\ldots$

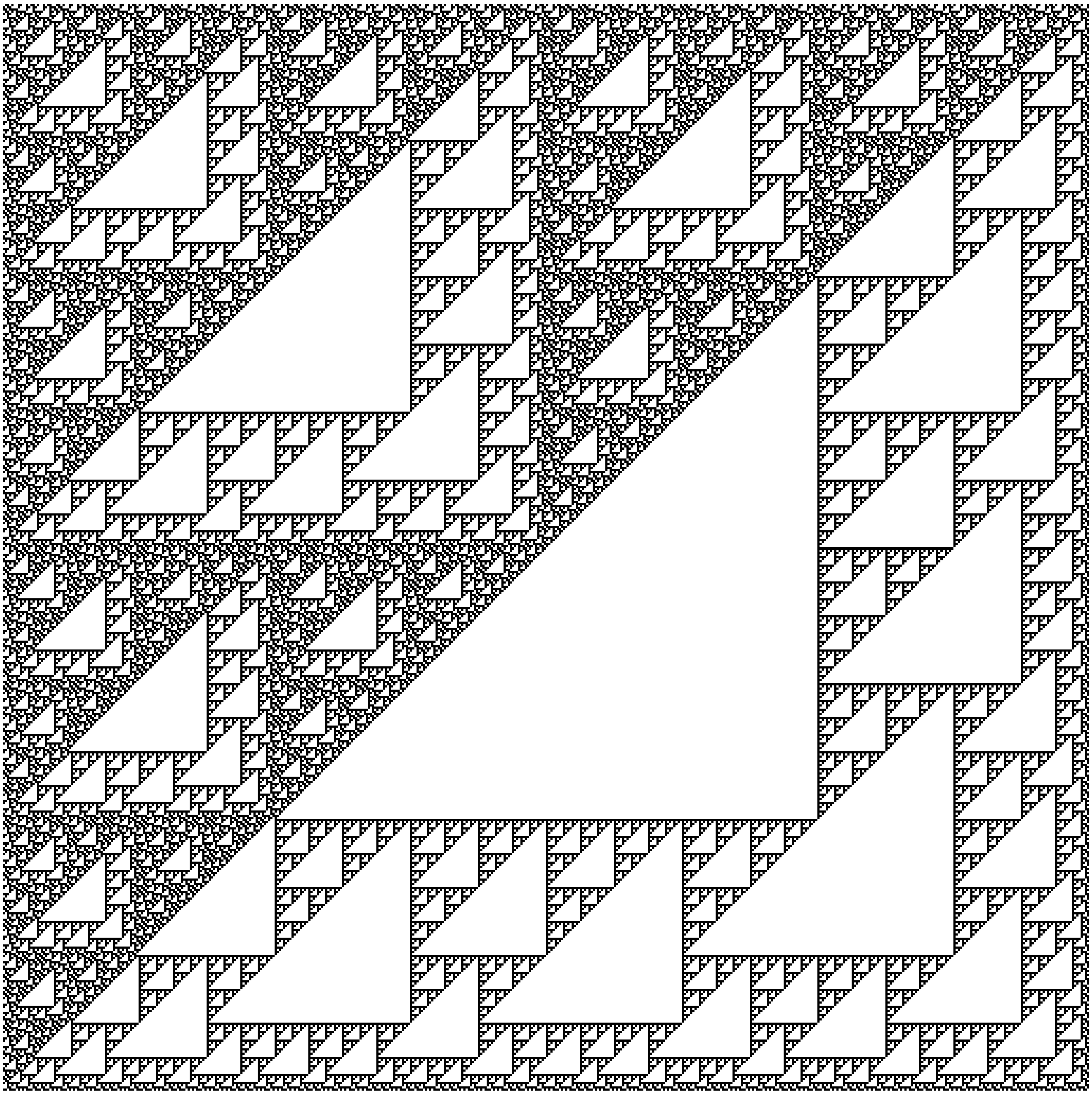
$t_n = s_2(n) \mod 2$

$[s_2(n) := \#\{i \,|\, a_i = 1, n = a_k 2^k + \cdots + a_0\}]$

$$\prod_{i=0}^{\infty}(1 - x^{2^i}) = \sum_{j=0}^{\infty}(-1)^{t_j} x^j$$

. . . .

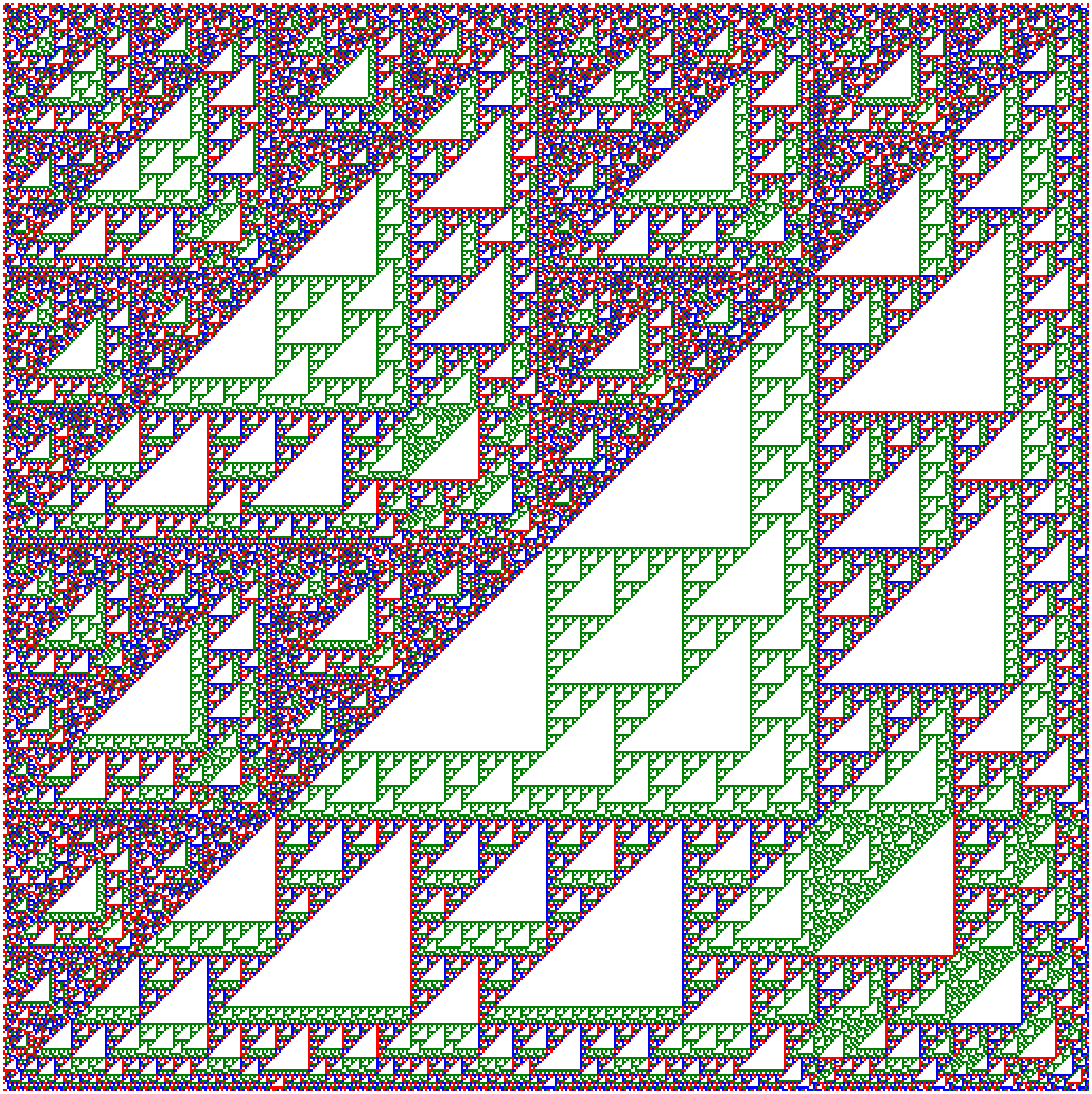
Pascal - Thue - Morse mod 2, 512×512



$(\mathbb{Z}/2\mathbb{Z}, x + z,$ Thue − Morse$)$

15 rules $4 \to 8$

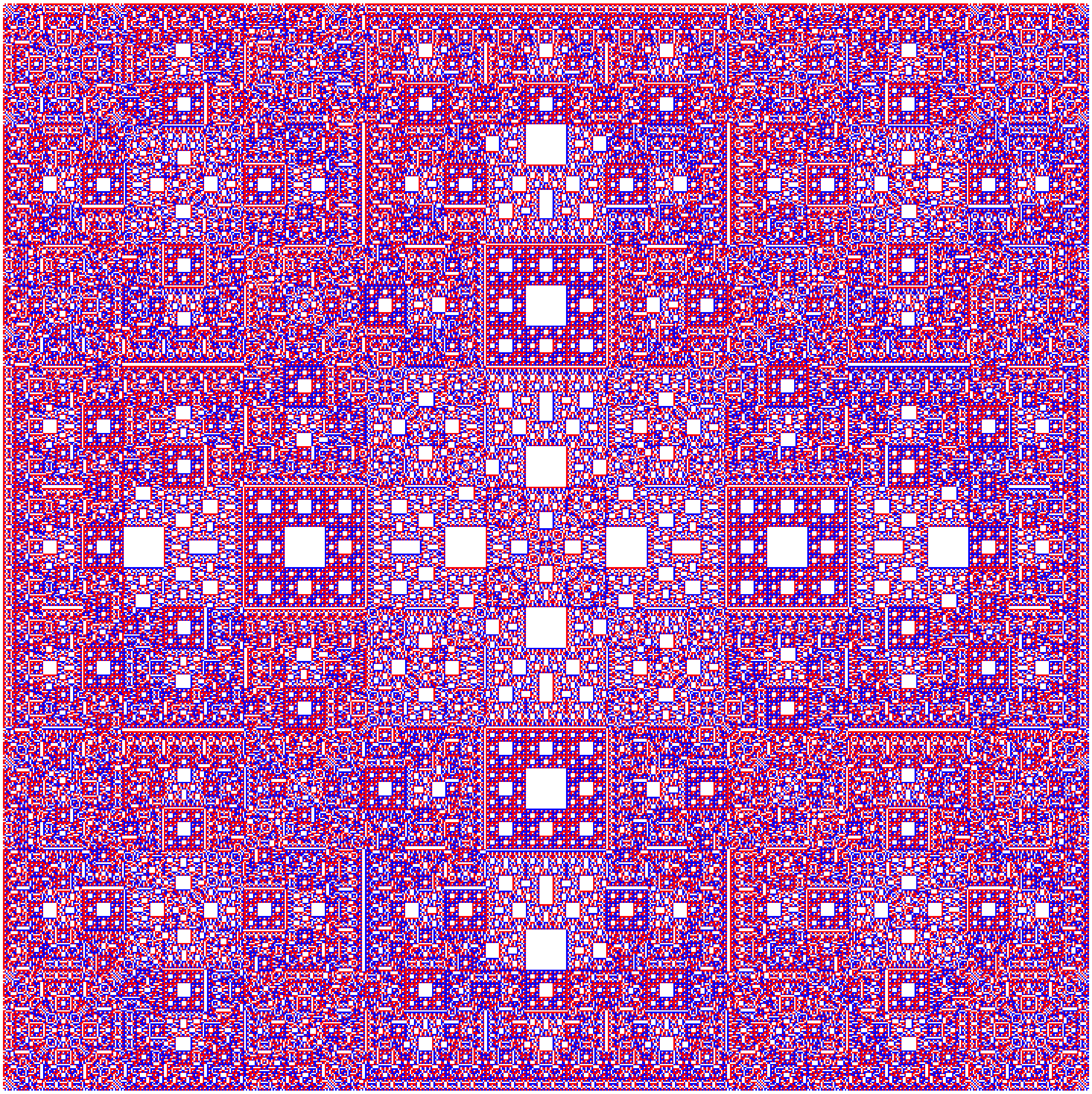
Pascal - Thue - Morse mod 4, 512×512



$(\mathbb{Z}/4\mathbb{Z}, x + z,$ Thue $-$ Morse$)$

284 rules $8 \to 16$

Arab Empire



$$(\mathbb{Z}/3\mathbb{Z}, x + y + z, 0 \to 010, 1 \to 111)$$

171 rules $3 \to 9$

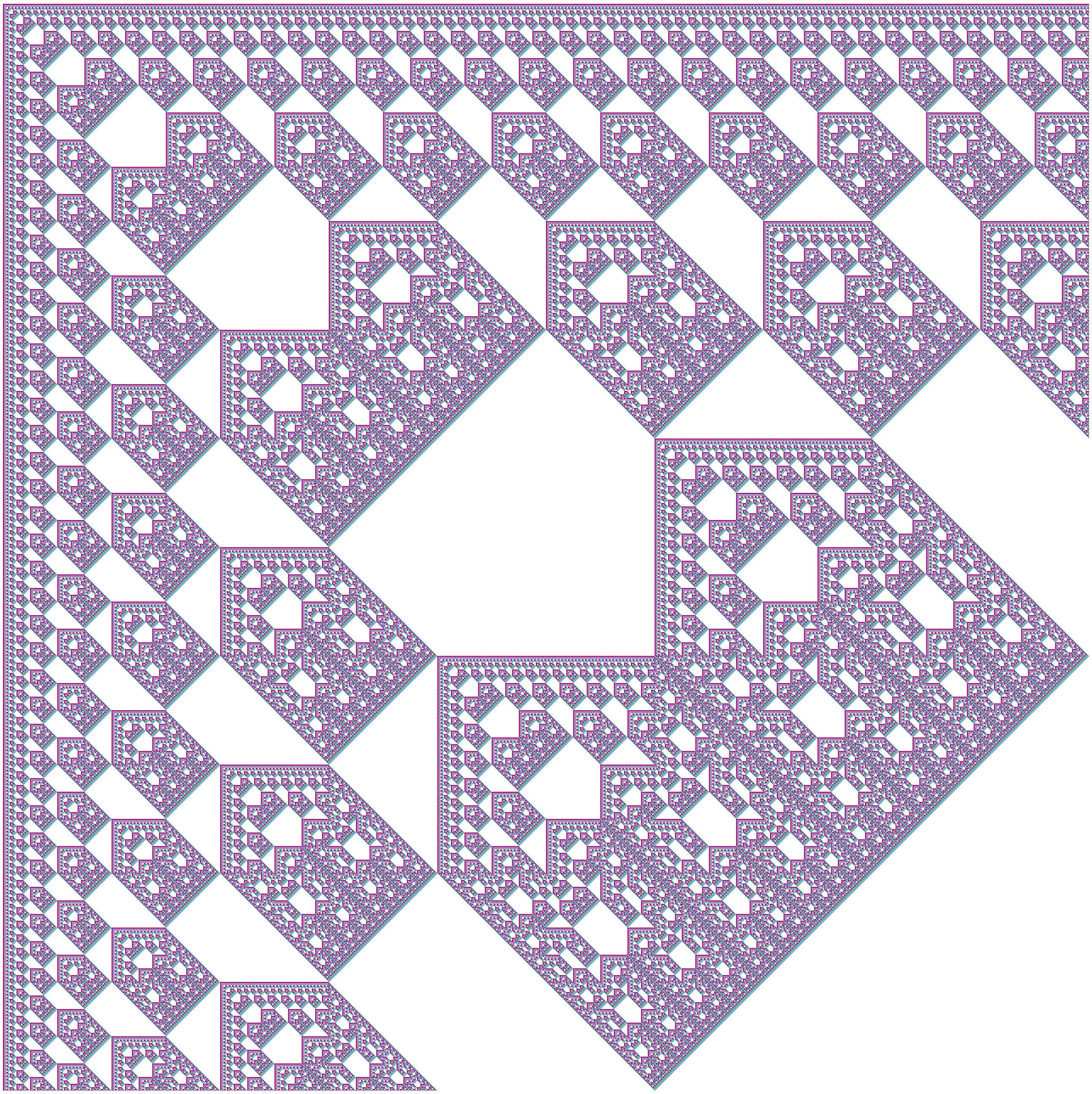
General Recurrence

$\vec{u}_1$, $\vec{u}_2$, ..., $\vec{u}_k > 0$ as elements of $\mathbb{Z}^n$ according to the lexicographic ordering.

$$f : A^k \to A$$

$$a(\vec{x}) = f(a(\vec{x} - \vec{u}_1), \ldots, a(\vec{x} - \vec{u}_k))$$

Twin Peaks, 2560×2560.

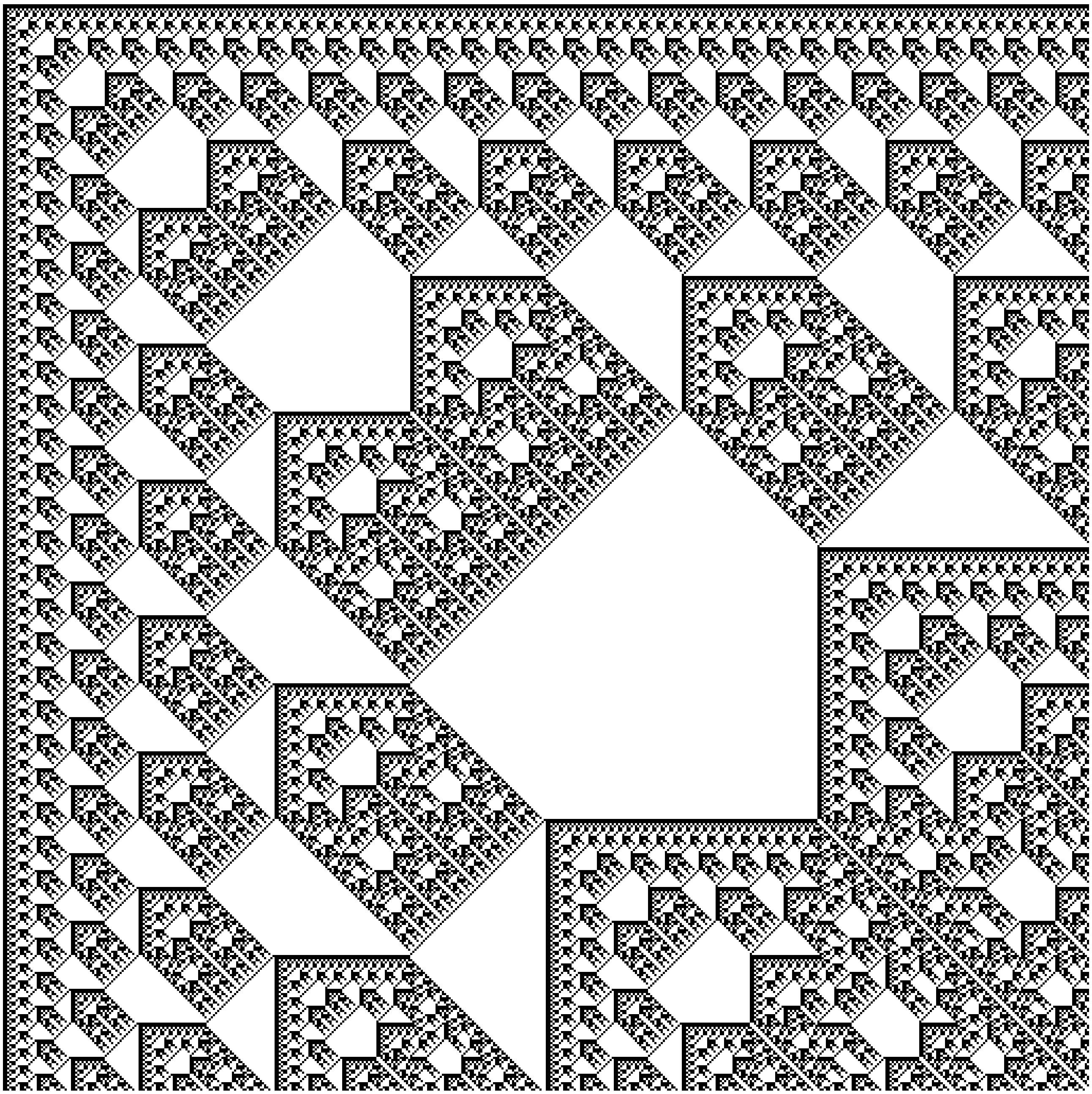


$$\mathbb{F}_4 = \{0, 1, \epsilon, \epsilon^2 = \epsilon + 1\} = \{0, 1, 2, 3\}$$

$$(\mathbb{F}_4, y + \epsilon(x + z) + \epsilon^2(x^2 + y^2 + z^2), 1)$$

15 rules $2 \to 4$

Twin Peaks

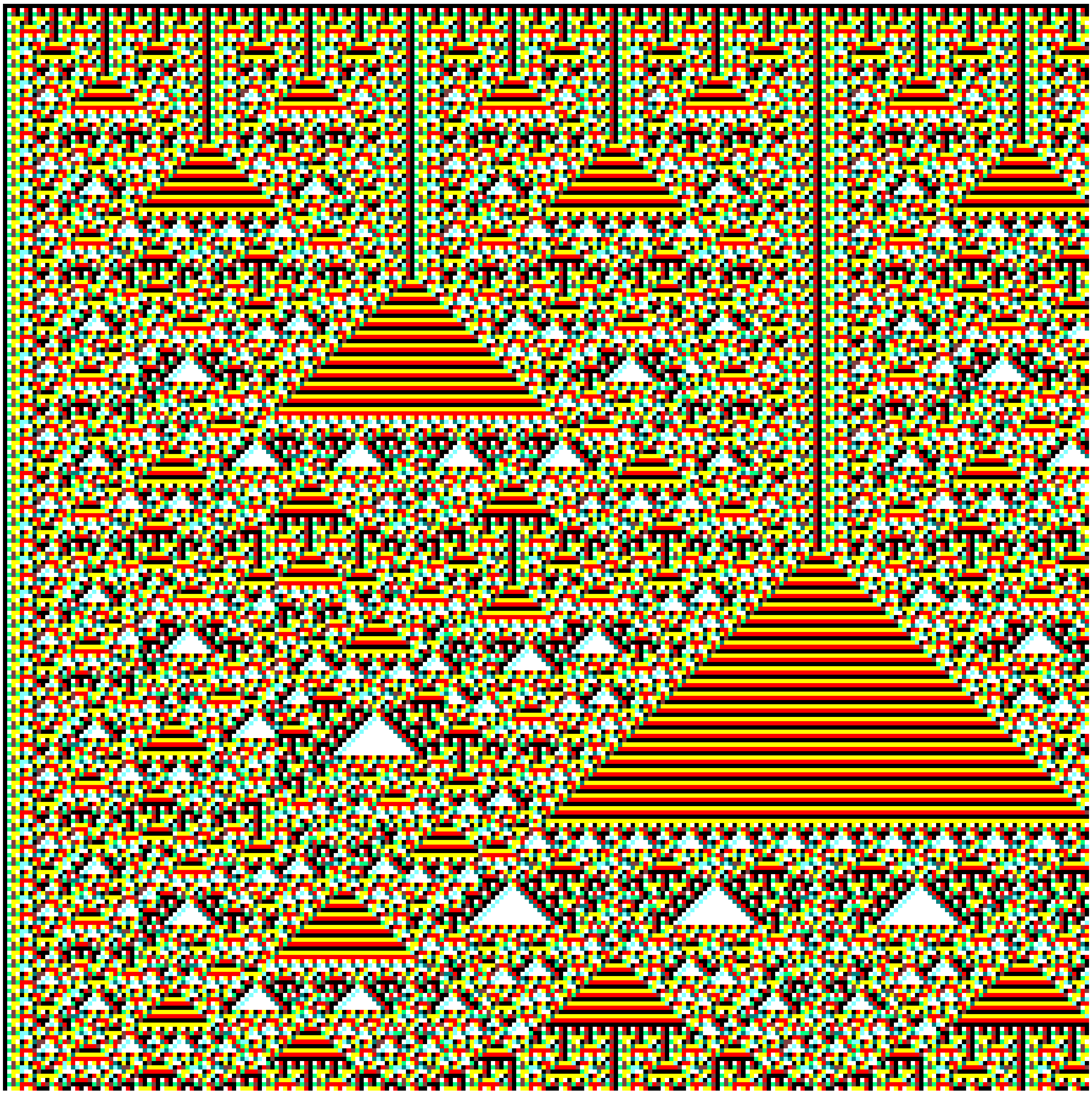


$(\mathbb{F}_2, x + y + z + t, 1, 1, 1, 1)$

$(0, 1)$, $(1, 2)$, $(2, 1)$, $(1, 0)$

15 rules $4 \rightarrow 8$

Lamps, Vincent van Gogh.

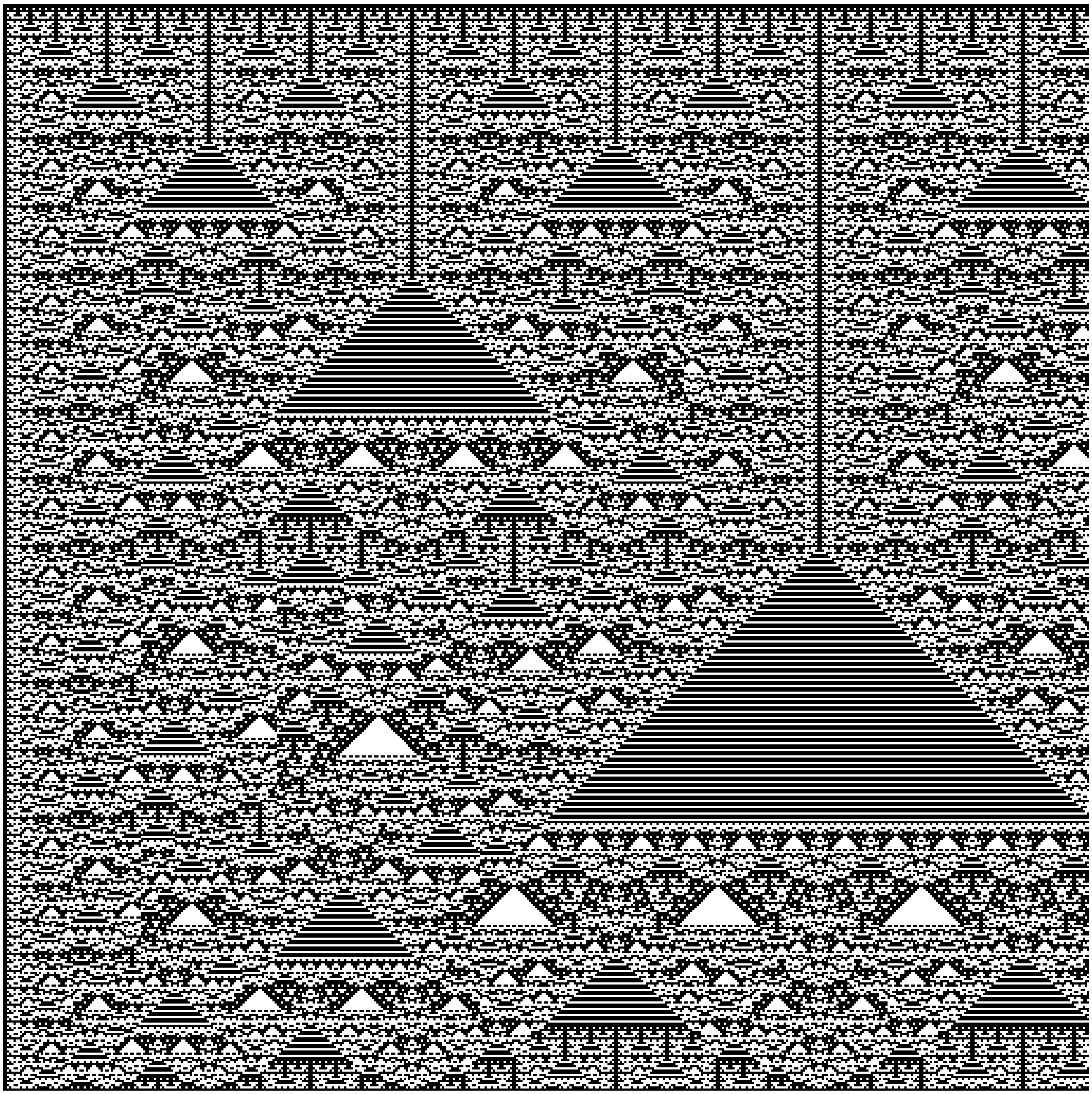


$$(M_2(\mathbb{F}_2), Yx + uy + cz, I)$$

Y, u, c, I constants

112 rules $2 \to 4$

Lamps, Vincent van Gogh.



$$(\mathbb{F}_2, x + y + z + t + u + v, 1, 1, 1, 1)$$

(0, 1), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1)

112 rules $4 \to 8$